A NEW SYSTEM OF MATHEMATICS.

CONTAINING

PLANE GEOMETRY;

GENERAL INVESTIGATION OF AREAS, SURFACES, AND SOLIDS;

GREATEST and LEAST QUANTITIES;

TRIGONOMETRY; LOGARITHMS;

MOTION,

UNIFORM, COMPOUND,
ACCELERATED, RETARDED,
PROJECTILES, APPLICATION.

To which is prefixed
The FIRST PRINCIPLES of ALGEBRA,
by way of INTRODUCTION.

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WILLIAM Duke of GLOUCESTER.

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TO

HIS ROYAL HIGHNESS

WILLIAM

DUKE OF GLOUCESTER,

THIS WORK

Is moſt humbly Dedicated,

BY

His Royal Highness's

Moſt devoted, moſt faithful,

and obedient Servant,

JOHN MULLER.

PREFACE.

THE design of this ſhort treatiſe, is to reduce the principal parts of the extenſive ſcience of Mathematics into ſo narrow a compaſs as to contain no more than what is abſolutely neceſſary to be known, with reſpect to practice in the different uſeful arts of life, to which mathematical knowledge may conveniently be applied; and hereby ſave both time and expence, as well as prevent that diſguſt occaſioned to many ſtudents, from a tedious round of intricate, and at the ſame time uſeleſs, ſpeculations.

If we conſider the great number of books in the various branches of Mathematics, which, according to the common courſe of inſtruction, a beginner is obliged to learn, before he can arrive at thoſe practical parts which his particular ſtation of life requires him to underſtand, it will not be ſurpriſing, that there are ſo few, who have patience to go through with ſo much application. To learn the elements of Euclid only, is, of itſelf, an arduous undertaking, and the greater part have ſeldom any inclination to go ſo far; even if they have time and reſolution enough to do it, they would then have learned but very little of what is neceſſary in practice.

For theſe reaſons, it ſeems requiſite to relieve ſtudents from ſuch burdenſome taſks, by cutting off all ſuperfluous ſpeculations, and retaining no more than what directly tends to inſtruction

in that practical knowledge required in various professions.

To proceed with order, I begin with all the necessary propositions of Plane Geometry, excluding all those which serve only as steps to prove others, which are of no use here; then proceed to a General Investigation of Arithmetical Progressions; and from thence are deduced two very easy General Rules, whereby not only many useful arithmetical questions are solved, but likewise the contents of all geometrical figures, both plane and solid, are found, by means of the analogy here shewn to subsist between Arithmetic and Geometry, not taken notice of before.

As there are but three plane figures, viz. the triangle, parallelogram, and circle; and but three regular solids, viz. the pyramid, prism, and sphere; so the contents of all plane figures, excepting the circle, are found by one single expression: in the same manner, all solids, excepting the sphere, are likewise found by a single expression.

My treating the doctrine of Proportions, will no doubt appear imperfect in respect to Incommensurables; but their nature is such, as can no otherwise be explained than by approximation; nor has Euclid been able, by all his subtility of reasoning, to explain them otherwise, as may be seen in prop. II. book XII.

The reader is supposed to understand the common rules of Algebra; but for the sake of beginners, who have not learned them before, we have inserted them here, by way of Introduction, and freed from obscurity some of the rules which others have not properly explained.

INTRO-

INTRODUCTION.

§. 1. ALGEBRA is the ſcience of Computation; it proceeds by rules and operations, ſimilar to thoſe in common Arithmetic, but more general and extenſive; it is grounded upon the ſame principles as Geometry, and therefore, is no leſs clear and convincing; it is likewiſe more general than Geometry, inaſmuch as its rules comprehend both Arithmetic and Geometry, including both ſciences in one: its great excellency, is to diſcover ſuch truths, by inveſtigation, as would in vain be ſought for by any other means. In Geometry, truth muſt be known before it can be demonſtrated; whereas it is diſcovered in Algebra, and the diſcovery is more advantageous than demonſtration, becauſe it leads the reader to find others not known to him before.

To proceed with method, in ſo extenſive a ſcience, it is neceſſary to explain the principles upon which it is built, in as clear and conciſe a manner as the ſubject will admit; and which muſt be well remembered, in order to underſtand perfectly what follows.

§. 2. Quantity being whatever may be meaſured, numbered, or eſtimated; numbers are nothing but certain ſigns or figures, invented to repreſent the order or relation between quantities of the ſame kind; ſo that whatever quantity is referred to unity, the numbers 2, 3, 4, denote twice, thrice, four times that quantity.

§. 3. Any part of a quantity may be referred to unity; as for example, a ſecond, minute, or hour of time; an inch, foot, or yard of length; a ſquare, or a cubic inch, foot or yard, of ſurfaces or ſolids.

§. 4. In Algebra, theſe relations are expreſſed in a more general manner, by letters, *a*, *b*, *c*, *d*; ſo that whatever quantity is referred to unity, *a* or *b* denotes as

many

many times that quantity, as there are units suppofed to be in a or b, and not the quantities themfelves, as they are commonly confidered; the values are always known in the application to queftions; but not neceffary in the rules of Addition, Subtraction, Multiplication, and Divifion, becaufe thefe rules are general, and not confined to any particular number.

§. 5. As thefe rules cannot be actually performed in Algebra, as in common Arithmetic, certain figns are ufed to indicate them; thus the fign + *plus* denotes addition, and the fign − *minus*, fubtraction: for example, if two numbers a and b are to be added, we write them $a+b$, but when b is to be fubtracted from a, we write $a-b$; and when feveral numbers are connected by thefe figns, as $a+b-c+d$, they ferve to fhew which are to be added, or fubtracted; that is, if a is 6, b 8, c 7, and d 5, then will $a+b-c+d$, become $6+8-7+5$ or 12, becaufe 6, 8, 5, are to be added, and 7 fubtracted from their fum 19.

The ſign + or −, belongs always to the number it precedes: thus in $a+b$, or $a-b$, the ſigns belong to b, and when a number has no ſign, it is ſuppoſed to have the ſign +; thus a, b, or $+a$, $+b$, mean the ſame thing.

§. 6. If a number $a+b$, is equal to another d, we write $a+b=d$; if a is greater than b, we write $a>b$, or if a is leſs than c, $a<c$, the greateſt ſtands always at the opening of the angle.

§. 7. Numbers preceded by the ſign +, are ſaid to be poſitive, and negative when preceded by the ſign −; thus a or $+b$, is a poſitive number, and $-b$, a negative one; they are both real, but oppoſite in their affection, the one increaſing, and the other diminiſhing the number to which they are joined. Though $+a$, $-a$, are equal as to their value, yet as their ſigns differ, we do not ſuppoſe that $+a=-a$, but $a-a=0$, becauſe to infer equality, they muſt not only be equal in value, but likewiſe in their affections.

§. 8.

§. 8. Equal numbers are always expreſſed by the ſame letters, and unequal ones, by different ones. The number of times that a number repreſented by a letter is taken, muſt be joined to it, as $3a$, $5b$ means, that the number a muſt be taken three times, and b five times; thus if a denotes 20 pounds, and b 10 yards; then 3 times 20 is 60 pounds, and 5 times 10 is 50 yards. All numbers ſtanding before letters, as here 3, 5, are called co-efficients; and when they have no co-efficient, unity is underſtood: thus $1a$, or $1b$, is the ſame as a or b.

§. 9. When ſeveral numbers, repreſented by letters, are to be multiplied together, they are joined as in a word; thus to multiply a by b, we write ab; and to multiply $3a$, b, c, we write $3abc$. Sometimes the ſign $\times$, is uſed to repreſent multiplication, eſpecially when ſeveral letters are joined together by the ſigns $+$ and $-$: for example; when $a+b$ is to be multiplied by $c+d$, we

write $\overline{a+b} \times \overline{c+d}$, with a ſtroke over the factors.

§. 10. When a number denoted by a letter, is to be multiplied by itſelf ſeveral times, the letter is ſet down with the number of factors above it; as *aa*, *aaa*, *aaaa*, are wrote a^2, a^3, a^4; theſe numbers 2, 3, 4, are called *indices* or *exponents*; moreover, a^2 is called the ſquare of a; a^3 the cube, and a^4 the fourth power.

§. 11. A number is ſaid to conſiſt of as many terms, as there are parts joined together by the ſigns + or −: thus $a+b$, conſiſt of two terms, and is called a *binomial*; $a+b+c$ of three terms, and is called a *trinomial*; a ſingle number a, or abc, is that which has but one term.

§. 12. Numbers are ſaid to be like, when they are expreſſed by the ſame letters, without conſidering their ſigns or numeral co-efficients; ſuch as $+4a$, and $-5a$, or as $3abc$, and $-4abc$.

ADDITION,

§. 13. *Is performed by connecting the several numbers together, with their proper signs.*

To	$+a$	$+3a$	$2a+3c$
add	$+3b$	$-3c$	$+4d$
sum	$a+3b$	$3a-3c$	$2a+3c+4d$

This rule is evident from the definition of the signs, §. 6, 7; for to add a negative number, means no more than to subtract its positive value.

Thus cash is positive, and debt negative; and therefore the real possession is the sum of all the cash, when all the debts are taken from it.

There are two cases in which Addition may be shortened.

CASE I.

§. 14. When numbers are like, and have like signs, *add the co-efficients, and prefix their sum to one of the numbers, with its sign.*

To

To	$+4a$	$-7ab$	$a+b$
add	$+3a$	$-3ab$	$3a+7b$
Sum	$+7a$	$-10ab$	$4a+8b.$

This rule is evident; for $4a$ and $3a$, makes $7a$, whatever number a may be; the ſame $-7ab$, and $-3ab$, make $-10ab$: in general, the ſum of all the co-efficients, of like numbers, and like ſigns, will always be the co-efficient of the ſum.

CASE II.

§. 15. If like numbers have different ſigns, *ſubtract the leaſt co-efficient from the greateſt, and prefix the difference to one of the numbers, with the ſign of the greateſt.*

To	$+6a$	$-7b$	$-ab$	$+4a-3b$
add	$-2a$	$+3b$	$+3ab$	$-2a+4b$
Sum	$+4a$	$-4b$	$+2ab$	$+2a+b.$

This is evident; for the leaſt number being taken from the greateſt, the difference muſt have the ſign of the greateſt.

When many like numbers, with different ſigns, are to be added, it will be con-

convenient to add firſt all thoſe with like ſigns, and then proceed as before.

$$\begin{array}{r} +5a-7b \\ -3a+4b \\ -2a+3b \\ +4a-5b \\ \hline \end{array}$$

The ſum of all the poſitives	$+9a+7b$
The ſum of all the negatives	$-5a-12b$
Total	$4a-5b.$

Obſerve, that all like numbers muſt be added, tho' they do not ſtand under each other; it is indifferent how they are placed, but are generally placed under each other for conveniency.

SUBTRACTION,

§. 16. *Is performed by changing the ſigns of the numbers to be ſubtracted into their contrary, then add them together as before.*

From	$+5a$	$-5ab$	$+6ac+3ad-5ab$
ſubtr.	$+3a$	$+3ab$	$+3ac-ad+2ab$
Diff.	$+2a$	$-8ab$	$3ac+4ad-7ab.$

For

For to ſubtract $+3a$ from $+5a$, change the ſign of $+3a$, and then $+5a-3a$, or $+2a$: to ſubtract $+3ab$, from $-5ab$, change the ſign of $+3ab$, which gives $-5ab-3ab$, or $-8ab$; and to ſubtract $+3ac-ad+2ab$, is the ſame thing as to add $-3ac+ad-2ab$.

Subtraction in Algebra, is proved by Addition, as in common Arithmetic. For the difference $+2a$, added to $+3a$, gives $+5a$; the difference $-8ab$, added to $+3ab$, gives $-5ab$; and $3ac+4ad-7ab$ added to $+3ac-ad+2ab$, gives $+6ac+3ad-5ab$.

MULTIPLICATION,

§. 17. *Is performed, by multiplying all the parts of one factor one after another, by all the parts of the other; and if the ſigns of the parts are like, the product muſt be poſitive, and negative if the ſigns are different.*

Mult.	$+a$	$-a$	$-a$
by	$+b$	$+b$	$-b$
Prod.	$+ab$	$-ab$	$+ab$

For

For to multiply a positive number a, by a positive one b, it is plain that a must be repeated as often as there are units in b, and their product is $+ab$.

To multiply a negative number $-a$, by a positive one $+b$; it is no less evident that $-a$ must be repeated as often as there are units in $+b$; and since a negative number repeated ever so often, remains negative; therefore, a negative number $-a$, multiplied by a positive one $+b$, gives $-ab$ for the product. But if a negative number $-a$, be multiplied by a negative number $-b$; then by the nature of the signs, $-a$ is to be repeated with a contrary sign as often as there are units in the multiplier b: that is, it is to be subtracted once, twice, thrice or b times: but to subtract $-a$ once, gives $+a$; twice $+2a$; thrice $+3a$; b times $+ab$.

This may be illustrated by some easy familiar examples, as follows:

Suppose a person wins b times a sum a; his total gain will be ab; if he loses b times the sum a, his total loss will be

$-ab$,

$-ab$, as oppoſite to his gain: but if he has loſt b times the ſum $-a$, and pays off as many times this ſum, as there are units in $-b$; it is plain, that the whole payment muſt be $+ab$, ſince he is richer by that ſum, than he was before: for to pay off the ſum $-a$ once, he gets $+a$; twice, $+2a$; and b times he gets $+ab$. This being plain in one ſpecies of quantity, muſt be ſo in all others.

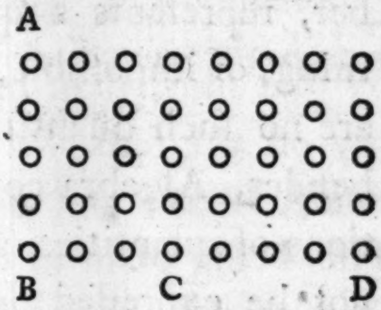
Mult.	$+2a$	$-5cd$	$-7bc$
by	$+3b$	$+4a$	$-4a$
Prod.	$+6ab$	$-20acd$	$+28abc$.

For a multiplied by $3b$, muſt be three times more than the product ab, becauſe the factor $3b$, is triple the factor b; and the product of $2a$ by $3b$, muſt be double that of a by $3b$; ſince $2a$ is double a. The product of $5cd$, by $4a$, muſt be four times the product $5acd$, ſince $4a$ is quadruple of a, that is, it muſt be $-20acd$: and in general,

§. 18. *Whatever numbers the co-efficients of the factors may be, their product will always*

ways be the co-efficient of the product of the factors.

It is not material in what order the letters of a product ſtand: for if the marks or counters in the line, AB, denote the units of one factor, and the marks in the line BD, the units of the other: then it is



plain that all the units in the factor AB, multiplied by the number of units in the factor BD, give the ſame product as all the units in the factor BD, multiplied by the units in the factor AB.

Hence, ab or ba, and 20×6, or 6×20, is the ſame thing, as well as abc, and acb, or bac: for $ab\times c$, or $ba\times c$, and $c\times ba$ is the ſame thing.

§. 19. The product of any two numbers AB, BD, is equal to the ſum of all the products of one of them AB, multiplied by all the parts BC, CD, of the other, divided any how; that is, a multiplied by $b+c+d$, gives $ab + ac + ad$. Again BD − CD, mul-

multiplied by AB, is equal to the product of BC, multiplied by AB; that is, $a-b$ multiplied by c gives $ac-cb$.

To pretend that a ſingle negative number, repreſents a quantity leſs than nothing, or impoſſible, is abſurd; ſince there are no ſuch quantities exiſting, by §. 2: beſides, Algebra conſiders only the relations of quantities, which when they cannot be expreſſed by any number, ſhews the ſuppoſition to be erroneous. The reaſons Thomas Simpſon gives, in pages 24, 25, to ſupport this abſurdity, are quite confuſed; ſometimes Algebra conſiders only magnitude, and afterwards abſtract numbers: and to ſhew it, he gives an example, which proves no more than that there are two caſes, viz. to find the perpendicular, when either the ſum or difference of the hypothenuſe, and one ſide of a right angled triangle is given, with the other ſide: ſo that what he took for an impoſſibility, is no more than another caſe of the ſame queſtion. The ſign has been confounded with the number; ſince what-

whatever number is denoted by a, it can never be but that number let its ſign be what it will. Amongſt many examples, the negative logarithms, which are as often uſed as the poſitive ones, is a convincing proof, that negative ſingle numbers are as uſeful as poſitive ones.

N. B. In Algebra, we always begin Multiplication at the left and go to the right, contrary to what is practiſed in Arithmetic, although it is indifferent which way it is done.

$$\begin{array}{lll} \text{Mult.} & a+b & a+b \\ \text{by} & \underline{a+b} & \underline{a-b} \\ & aa+ab & aa+ab \\ & \underline{\quad +ab+bb} & \underline{\quad -ab-bb} \\ \text{prod.} & aa+2ab+bb & aa \cdot\cdot\, 0-bb. \end{array}$$

For $a+b$ multiplied by a, gives $aa+ab$, and $a+b$ multiplied by $+b$, gives $ab+bb$; theſe two products being added, gives $aa+2ab+bb$. Now $a+b$ multiplied by a, gives $aa+ab$ as before; but $a+b$ multiplied by $-b$, gives $-ab-bb$, which added to $aa+ab$ gives $aa-bb$, becauſe $ab-ab=0$, by §. 7.

Mult.	$a-b$	$6-3$
by	$a-b$	$6-3$
	$aa-ab$	$36-18$
	$-ab+bb$	$-18+9$
prod.	$aa-2ab+bb$	$36-36+9=9.$

Because $a-b$ multiplied by a, gives $aa-ab$; but a being greater than $a-b$ by b, the product $aa-ab$ is too much by the product $ab-bb$ of b by $a-b$; which therefore must be subtracted from $aa-ab$, to get $aa-2ab+bb$ for the product of $a-b$ by $a-b$: this shews, that $-b$ multiplied by $-b$, must give $+bb$; but by no means proves, that $-$ by $-$ gives $+$; which must be done from the nature of the signs themselves, as we have shewn.

When several factors are to be multiplied together, multiply any two and the product, by the third, and continue so to the last: but when different powers of the same number are to be multiplied together, then the index of the product, is equal to the sum of those of the factors. For example: $aa=a^2$, $a\times a^2=a^3$, $a^2\times a^3=a^5$, or in general, $a^n\times a^m=a^{n+m}$. When a compound number is to be raised to any

5 power,

power, a line is drawn over it with the index at the end. Thus the square of $a+b$ is marked $\overline{a+b}|^2$, the cube $\overline{a+b}|^3$. Hence $a+b$ multiplied by $\overline{a+b}|^2$, gives $\overline{a+b}|^3$, and $\overline{a+b}|^2$ multiplied by $\overline{a+b}|^3$, gives $\overline{a+b}|^5$.

DIVISION,

§. 20. *Is performed, by finding the number of times that the divisor is contained in the dividend; if both have the same sign, the quotient must be positive, and negative if the signs are different.*

Thus $+ab$ divided by $+a$, gives $+b$ for the quotient; $-3ab$ divided by $+a$, gives $-3b$; $-6abc$, divided by $-2c$, gives $+3ab$.

Division is proved by Multiplication; therefore, the quotient $+b$ multiplied by the divisor $+a$, gives $+ab$; the quotient $-3b$, multiplied by the divisor $+a$, gives $-3ab$; and $-2c$, multiplied by $+3ab$, gives $-6abc$.

If the divisor is not contained exactly in the dividend, divide both by a com-

mon divisor, if they have any, and set down what is left, with a stroke between them, as follows:

divid.	a	$6cd$	$+7abc$	$-4aab$
by	b	$3b$	$-2bd$	$-2cd$
quot.	$\frac{a}{b}$	$\frac{2cd}{b}$	$\frac{-7ac}{2d}$	$\frac{+2aab}{cd}$

For $\frac{a}{b}$ multiplied by b, gives $\frac{ab}{b}$ or a; $\frac{2cd}{b}$ multiplied by $3b$, gives $\frac{6cdb}{b}$ or $6cd$; $-\frac{7ac}{2d}$ multiplied by $-2bd$, gives $+\frac{14abcd}{2d}$ or $+7abc$; and $+\frac{2aab}{cd}$ multiplied by $-2cd$, gives $-\frac{4aabcd}{cd}$ or $-4aab$: since a number multiplied and divided by the same number, neither increases nor diminishes its value.

If the divisor and dividend are compound, *set them down as in common Arithmetic, and divide one part of the dividend after another as usual.*

$$\text{Divisor } 2a)\ 6ab + 10ac\ (3b + 5c \text{ quot.}$$
$$\begin{array}{cc} \underline{6ab} & \underline{10ac} \\ 0 & 0 \end{array}$$

For

For $6ab$ divided by $2a$, gives $3b$ for the firſt term of the quotient; and the product $6ab$ of the quotient $3b$ and the diviſor $2a$ being ſubtracted, leaves $+10ac$; this divided by $2a$, gives $+5c$ for the ſecond term of the quotient, and the product of $5c$ by $2a$ being ſubtracted, leaves o.

Since the quotient $3b+5c$ multiplied by $2a$, gives $6ab+10ac$, for the dividend.

$$\begin{array}{l} \text{diviſor } a-b)\ aa-2ab+bb\ (a-b \text{ quot.} \\ \qquad\qquad\quad \underline{aa-ab} \\ \qquad\qquad\quad o-ab+bb \\ \qquad\qquad\quad\ \ \underline{-ab+bb} \\ \qquad\qquad\qquad\ o \quad o \end{array}$$

For aa divided by a, gives a for the firſt part of the quotient; and the product $aa-ab$ of the quotient a and the diviſor $a-b$, being ſubtracted leaves $-ab+bb$; which divided by a, gives $-b$ for the ſecond part of the quotient; this multiplied by the diviſor $a-b$, and the product $-ab+bb$ ſubtracted, leaves o.

If any part of the product of the diviſor multiplied by the quotient, cannot be ſubtracted, ſet it down as a part of the remainder with a contrary ſign.

$$\begin{array}{l} \text{Divisor } a-b)\ aa-bb\ (a+b \text{ quot.} \\ \qquad\qquad\quad \underline{aa-ab} \\ \qquad\qquad\quad 0+ab-bb \\ \qquad\qquad\qquad \underline{+ab-bb} \\ \qquad\qquad\qquad\ \ 0 \quad 0 \end{array}$$

When the divisor and dividend have a common divisor, divide both by it and proceed as before: thus, if $a^3bc-10abcxx+3bcx^3$, is to be divided by $abc-3bcx$, strike out bc in both, then the division will stand.

$$\begin{array}{l} a-3x)\ a^3-10axx+3x^3\ (aa+3ax-xx \\ \qquad\quad \underline{a^3-\ 3aax} \\ \qquad\quad 0+\ 3aax-10axx \\ \qquad\quad \underline{+\ 3aax-\ 9axx} \\ \qquad\quad 0 \qquad\quad -axx+3x^3 \\ \qquad\qquad\qquad\ \ \underline{-axx+3x^3} \\ \qquad\qquad\qquad\quad\ 0 \quad 0 \end{array}$$

When the divisor is not contained exactly in the dividend, the remainder is set down with the divisor under it, as a part of the quotient. Thus $aa+ab+bb$, divided by $a+b$, gives $a+\frac{b}{a+b}$ for the quotient.

It is necessary in some cases, to carry on the division so far as that the quotient becomes an infinite series: as here, where a divided by $1+x$, gives

$1+x$

$$
\begin{array}{l}
1+x)\,a\,(a-ax+axx-ax^3+,\ \&c. \\
\quad \underline{a+ax} \\
\quad 0-ax \\
\quad \underline{\quad -ax-axx} \\
\qquad\quad 0\ +axx \\
\qquad\quad \underline{+axx+ax^3} \\
\qquad\qquad 0\ \ -ax^3
\end{array}
$$

SQUARE ROOT.

§. 21. The ſquare root of any number, is ſuch as when it is multiplied by itſelf, ſhall give the propoſed number; for example: the ſquare root of aa is a, ſince a multiplied by a gives aa: the ſquare root of 9 is 3, for 3 multiplied by 3 gives 9; the ſquare root of 144 is 12, ſince 12 multiplied by 12 gives 144.

§. 22. *To find the ſquare root of* $aa+2ab+bb$.

The ſquare root of the firſt term aa, is a, whoſe ſquare aa ſubtracted leaves $2ab+bb$; now if the root found a be doubled and the remainder divided by $2a$, we get b for the ſecond term of the root; and if this term be added to the diviſor $2a$ and the ſum $2a+b$ multiplied

by b, gives $2ab+bb$, which being subtracted leaves o. Therefore $a+b$ is the root required; for $\overline{a+b}^2$, gives $aa+2ab+bb$.

$$\begin{array}{lll} \text{Operation} & aa+2ab+bb(a+b & \\ & \underline{aa} & \\ 2a) & \text{o} \quad 2ab+bb & \\ & \quad\ \ \underline{2ab+bb} & \\ & \quad\ \ \ \text{o} \quad \text{o} & \end{array}$$

§. 23. *To find the square root of* $aa+2ab+bb+2ac+2bc+cc$.

The root of aa is a, whose square aa being subtracted leaves $2ab+bb+$, &c. divide the remainder by $2a$ twice the root found, which gives b for the next term of the root; this being added to the divisor $2a$, and the sum $2a+b$ multiplied by b gives $2ab+bb$, and this subtracted leaves $2ac+2bc+cc$; this remainder being divided by $2a+2b$, twice the root already found gives c; c added to the divisor $2a+2b$ and multiplied by c gives $2ac+2bc+cc$; which when subtracted from the remainder leaves o. Therefore $a+b+c$ is the root required: since if this root be multiplied by itself it gives the square proposed.

Operation

$$
\begin{array}{lrl}
\text{Operation} & & aa+2ab+bb+2ac+2bc+cc(a+b+c \\
& & \underline{aa} \\
& 2a) & 0+2ab+bb \\
& & \underline{+2ab+bb} \\
& 2a+2b) & 0 \quad 0+2ac+2bc+cc \\
& & \qquad\quad \underline{2ac+2bc+cc} \\
& & 0 \quad 0 \quad 0
\end{array}
$$

GENERAL RULE.

Having found the rooot of the firſt term, to find any of the ſucceeding ones.

Divide always by twice the root already found, which gives the next term; this being added to the diviſor, multiply the ſum by that term, and ſubtract the product.

When the root cannot be found exactly, it is neceſſary to carry on the operation ſo far, as the law of continuation may be diſcovered; this appears by the following example, where the ſquare root of $1-xx$, is required.

$$
\begin{array}{rl}
1-xx) & 1-\frac{1}{2}xx-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8-, \&c. \\
2) & .-xx \\
 & \underline{-xx+\frac{1}{4}x^4} \\
2-xx) & .-\frac{1}{4}x^4 \\
 & \underline{-\frac{1}{4}x^4+\frac{1}{8}x^6+\frac{1}{64}x^8} \\
2-xx) & .-\frac{1}{8}x^6-\frac{1}{64}x^8 \\
 & \underline{\frac{1}{8}x^6+\frac{1}{16}x^8} \\
2) & ,-\frac{5x^8}{64}.
\end{array}
$$

Obſerve,

Obſerve, that when any part of the product cannot be ſubtracted, to place it to the remainder with a contrary ſign, the ſame as in Diviſion; and that after the two firſt terms of the root have been found, no more of the diviſor has been ſet down, than was ſufficient to find five terms, whereby the law of Continuation is diſcovered. For if $a, b, c, d, f, g,$ denote the numeral co-efficients; then $a = \frac{1}{2}$, $b = \frac{1}{4}a$, $c = \frac{3}{6}b$, $d = \frac{5}{8}c$, $f = \frac{7}{10}d$, $g = \frac{9}{12}f$, &c. ſo that the upper numbers are 1, 3, 5, 7, 9, 11, &c. and the under ones, 2, 4, 6, 8, 10, 12, &c.

The ſquare root of common numbers, is found by the ſame rule; *ſeparating them two by two, from the right to the left; then the root will conſiſt of as many figures as there are diviſions.*

For the root of 321489 is found, by ſeparating them as in the margin.

Then the neareſt ſquare under 32 is 25, whoſe root 5, is the firſt figure of the root, and its ſquare 25 ſubtracted from 32

```
     32,14,89(567
     25
  10)714
     636
 112)7889
     7889
     ....
```

leaves

leaves 7; to this join the two next figures 14, and divide 714 by 10, twice the root 5, which gives 6 for the next figure of the root; join this figure 6 to the divisor 10; multiply the sum 106 by 6; subtract the product 636 from 714 and there remains 78; to this join the two next figures 89; divide 7889 by 112, twice the root 56, which gives 7 for the next figure of the root; join this figure 7 to the divisor 112; multiply the sum 1127 by 7; subtract the product 7889 from 7889 and there remains 0. Hence, 567 is the root required, since this number multiplied by itself gives the square proposed.

If the root cannot be found exactly, it is sometimes necessary to carry the operation farther, which is done by annexing to the remainder twice as many cyphers as you want decimals; as in the example joined here; where the square root of 140 is required to two decimals; now as the square root of one is 1, and one subtracted from 1 leaves 0; divide the next two figures 40 by

```
        1,40(11.83
        1
    2) , 40
         21
   22)  1900
        1824
  236)..7600
        7089
remain   511
```

by 2, twice the first figure 1 of the root, which gives one for the next figure of the root; this figure joined to the divisor 2, multiply the sum 21 by 1; subtract the product 21 from 40, and there remains 19; if to this remainder two cyphers are joined, and the operation continued, by adding every time two cyphers, the root may be found to any number of decimals.

We separate the decimals from the whole numbers by a point, to distinguish the one from the other.

VULGAR FRACTIONS.

§. 24. A fraction consists of two parts, one placed over the other, as $\frac{2}{3}$ or $\frac{b}{a}$; the under one, 3 or *a*, shews in how many parts unity is divided, and is called the *denominator*; and the upper, 2 or *b*, how many of these parts the fraction contains, and is called the *numerator*: thus $\frac{2}{3}$ means two thirds of any thing, as of a yard, pound, &c. When the numerator is greater than the denominator, as $\frac{4}{3}$, or $\frac{7}{5}$, the fraction is called improper: Hence, any

any whole number may be called an improper fraction whose denominator is unity.

§. 25. To reduce a fraction to its lowest denomination.

Divide both its numerator and denominator by the greatest common divisor, if they have any; and if they have not, the fraction cannot be reduced.

The fraction $\frac{6}{12}$ divided by 6 gives $\frac{1}{2}$; the fraction $\frac{56}{240}$ divided by 8 gives $\frac{7}{30}$; $\frac{ab}{ac}$ divided by a gives $\frac{b}{c}$.

This rule is evident, since if you multiply and divide any number, by any but the same number, its value is neither increased nor diminished.

§. 26. To find the greatest common divisor of any two numbers.

Divide the greatest by the least, then the least by the remainder, and continue to divide always the last divisor by the last remainder, till you find a divisor that divides exactly, which will be the required one.

Example: Let 89046 and 2346, be the given numbers, divide the first by the last, divide the last 2346 by the remainder

 2244,

2244, and the divisor 2244 by the last remainder 102, which leaving 0 will be the divisor sought. For 2346 divided by 102 gives 23, and 89046 divided by 102 gives 873.

The demonstration of this rule depends on this, that the divisor of any number divides itself, and any multiple of that number, as likewise its equal, and any of its multiples.

§. 27. To reduce fractions under the same denomination.

Multiply all the numerators separately, by all the denominators excepting their own; then the products will be the numerators, and the product of all the denominators, the common denominator.

Let $\frac{2}{3}$, $\frac{5}{7}$, be the fractions; multiply the numerator 2 by the denominator 7, and the numerator 5 by the denominator 3, which gives the numerators 14, 15, and the product 21 of the denominators 3, 7, the denominator. For $\frac{2}{3}=\frac{14}{21}$, and $\frac{5}{7}=\frac{15}{21}$.

If $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{7}$, be the fractions, multiply the numerator 1 by the product 21 of the

the denominators 3, 7; multiply the numerator 2 by the product 14 of the denominators 2, 7, and multiply the numerator 4 by the product 6 of the denominators 2, 3, which gives the numerators 21, 28, 24, and the product 42 of all the denominators, the common denominator. For $\frac{1}{2} = \frac{21}{42}$, $\frac{2}{3} = \frac{28}{42}$, and $\frac{4}{7} = \frac{24}{42}$.

The fractions $\frac{b}{a}$, $\frac{d}{c}$, give $\frac{b}{a} = \frac{bc}{ac}$ and $\frac{d}{c} = \frac{ad}{ac}$; the fractions $\frac{b}{a}$, $\frac{d}{c}$, $\frac{f}{n}$, give $\frac{b}{a} = \frac{bcn}{acn}$, $\frac{d}{c} = \frac{adn}{acn}$, and $\frac{f}{n} = \frac{acf}{acn}$. This rule is evident, ſince multiplying the numerator and the denominator of a fraction, by the ſame number, neither increaſes or decreaſes its value.

§. 28. To add and ſubtract fractions.

Reduce them under the ſame denomination; then the ſum or difference of the numerators, divided by the common denominator, will be the fraction required.

Thus $\frac{2}{3} + \frac{3}{4} = \frac{8+9}{12}$, or $\frac{17}{12}$, is the ſum; $4 - \frac{2}{3} = \frac{12-2}{3}$, or $\frac{10}{3}$ the difference, and

$\frac{1}{2}$

$\frac{1}{2}+\frac{2}{3}-\frac{1}{8}=\frac{24+32-6}{48}$, or $\frac{50}{48}$, which when divided by 2, gives $\frac{25}{24}$.

This rule is evident, ſince like parts may be added and ſubtracted in the ſame manner as whole numbers; that is, thirds to thirds, fourths to fourths, &c.

§. 29. To multiply and divide fractions.

In multiplication, *the product of all the numerators, divided by the product of all the denominators, gives the product*; in diviſion, *invert the diviſor, then it becomes a multiplication*.

To multiply a by $\frac{b}{c}$, gives $\frac{ab}{c}$, to multiply $\frac{b}{a}$ by $\frac{d}{c}$, gives $\frac{bd}{ac}$. For the product ab is ſo much greater than $\frac{ab}{c}$, as the factor b is greater than $\frac{b}{c}$ by the nature of multiplication, and the product $\frac{bd}{a}$ is ſo much greater than $\frac{bd}{ac}$, as the factor d is greater than the factor $\frac{d}{c}$. The ſame thing is true in regard to any other fraction.

To

To divide a by $\frac{b}{c}$; then $a \times \frac{c}{b}$ or $\frac{ac}{b}$, is the quotient: $\frac{b}{a}$ divided by $\frac{c}{d}$, gives $\frac{b}{a} \times \frac{d}{c}$ or $\frac{bd}{ac}$.

If the denominators are divisible by the same number as the denominators, *divide as many of the one as of the other, and multiply the quotients.*

Thus $\frac{3}{4} \times \frac{2}{3}$, gives $\frac{1}{2} \times 1$ or $\frac{1}{2}$; $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$ gives $1 \times 1 \times \frac{1}{4}$ or $\frac{1}{4}$: for $\frac{b}{a} \times \frac{a}{c} \times \frac{c}{d}$, gives $\frac{abc}{acd}$ or $\frac{b}{d}$, when divided by ac.

The square root of a fraction is found, by extracting the root of both numerator and denominator separately: thus the square root of $\frac{25}{49}$ is $\frac{5}{7}$, and the square root of $\frac{bb}{aa}$ is $\frac{b}{a}$.

§. 30. To reduce a number into a fraction of a higher denomination: *divide it by the number of times the lower is contained in the unit of the higher.*

Thus pence divided by 12, give fractions of a shilling: shillings divided by 20, give fractions of a pound: as 3 pence

gives $\frac{3}{12}$ or $\frac{1}{4}$ of a ſhilling; 15 ſhillings divided by 20, gives $\frac{15}{20}$, or $\frac{3}{4}$ of a pound.

§. 31. To reduce fractions into whole numbers of a lower denomination; *multiply the numerator by the number of times that the lower is contained in the higher.*

Thus the fraction $\frac{1}{3}$ of a ſhilling, mulplied by 12, and divided by 3, gives 4 pence; the fraction $\frac{3}{4}$ of a pound, mulplied by 20, and divided by 4, 15 ſhillings.

§. 32. To reduce a fraction into decimals: *annex as many cyphers to the numerator as you want decimals, and divide by the denominator.*

Thus $\frac{7}{8}$ is reduced into decimals, by joining three cyphers to the numerator 7, and dividing 7000 by the denominator 8, which gives .875: if $\frac{1}{32}$, join five cyphers to the numerator, and divide by 32; then will .03125. when the diviſion is not exact: what remains after three or four decimals is neglected, as in the fraction $\frac{5}{12}$, by joining four cyphers, gives .4166, and there remains $\frac{2}{3}$, which is neglected.

Decimals are added and ſubtracted like whole numbers, obſerving only, *to place the points which ſeparate them from the whole*

whole numbers, under each other. They are likewiſe multiplied and divided as whole numbers, *by making as many decimals in the product, as there are in both factors*; and in the quotient, *as there are more in the dividend than in the diviſor.*

Thus .125 multiplied by 6.4 gives .8000, or .8 only: .025 multiplied by 3 gives .075; and .378 divided by 63 gives .006.

The value of a decimal fraction is found, *if multiplied by the number of times that the lower denomination is contained in the unit of the higher, and pointing off as many decimals as there were before.*

The decimal .875 of a pound, multiplied by 20, gives 17.5 ſhillings, or 17 ſhillings and 6 pence; and .03125 of a pound, multiplied by 20 gives .625 of a ſhilling; this multiplied by 12, gives 7.5 pence, and 5 multiplied by 4 gives 2.0 or 2 farthings.

To reduce decimals into common fractions: *Divide the decimals by unity, with as many cyphers annexed to it as there are decimals.*

Thus the decimal 1.5 divided by 10 gives $\frac{15}{10}$ or $\frac{3}{2}$ when divided by 5; and

.875 divided by 1000 gives $\frac{875}{1000}$ or $\frac{7}{8}$ when divided by 125.

Equations *and their* Reduction.

§. 33. The equality between two numbers, consisting of one or more terms, is called an *equation*, as $ax=bc$, or $a+x=b$.

In solving questions by Algebra, not only the known numbers are expressed by letters, but likewise the unknown or sought ones; and for distinction sake, the known numbers are always expressed by the initial letters, *a*, *b*, *c*, *d*, and the unknown or sought ones, by the final *u*, *x*, *y*, *z*.

It is from the relation of the known or given numbers to the unknown or sought ones, that equations are formed.

For example: to find a number x from which 4 being subtracted, the difference shall be equal to 6; we set down $x-4=6$. To find a number x whose two thirds shall be equal to 4, we set down $\frac{2}{3}x=4$: and to find a number x whose fourth part shall be equal to the difference of 5, from which one sixth part of the unknown number being subtracted, we get $\frac{1}{4}x=5-\frac{1}{6}x$.

The

The value of an unknown number is ſaid to be found, when it ſtands on one ſide of the equation with a poſitive ſign, freed from its co-efficient, if it has any, and all the known terms on the other. The ſeveral operations whereby the value of an unknown number is determined, are called reductions.

Reduction by Addition and Subtraction.

§. 34. If a known number ſtands on the ſame ſide as the unknown one, *tranſpoſe it to the other ſide with a contrary ſign.* This is no more than to add or ſubtract equals to or from equals, which does not deſtroy the equality.

Example: If $x-4=6$, we get $x=6+4$, by addition, or $x=10$, by contraction. If $x+b=a$, then $x=a-b$, by ſubtraction. If $6-x=2$; then $6-2=x$ by tranſpoſition, $4=x$ by contraction. For if we add 4 to both ſides of the equation $x-4=6$, then $x+4-4=6+4$, or $x=10$, becauſe $+4-4$ on the firſt ſide deſtroy one another: and if we ſubtract b from both ſides of the equation $x+b=a$, we get $x+b-b=a-b$, or $x=a-b$, becauſe $+b-b=0$.

Reduction

Reduction by Multiplication and Division.

§. 35. If the unknown number be multiplied or divided by a known one; strike it out, and divide or multiply all the other terms by it: this is no more than to divide or multiply equals by equals, which does not destroy the equality.

Example: If $2x=2a+4b$; then $x=a+2b$, by equal division. If $\frac{1}{2}x=a+c$; then $x=2a+2c$, by equal multiplication; since x multiplied and divided by the same number, neither increases nor decreases its value.

If $\frac{2}{3}x=4$; then $2x=12$ by equal multiplication, and $x=6$ by equal division. Lastly, if $\frac{1}{4}x=5-\frac{1}{6}x$; then multiplying by the denominators 4 and 6, we get $6x=120-4x$, and $10x=120$ by transposition and contraction, or $x=12$ by equal division.

We do not consider here the reduction of equations, when there are several unknown quantities contained in one or more equations, as being of no use in the rest of this work.

Reduction by the Roots and Powers.

§. 36. When the highest power of the unknown quantity in an equation is the square,

ſquare, it is called a *quadratic* equation; when that power is a cube, a cubic equation, &c.

The ſquare root of any number, as $a+b$, is marked $\sqrt{a+b}$. If $\sqrt{a-x}=c$; then by ſquaring both ſides of the equation, we get $a-x=cc$, and $a-cc=x$, by tranſpoſition. If $\sqrt{4x+5}=5$; by ſquaring both ſides, we get $4x+5=25$, or $4x=25-5=20$ by tranſpoſition, and $x=5$ by diviſion.

§. 37. All quadratic equations may be repreſented by this one form $xx+2ax=bc$, wherein the ſecond term $2ax$ may be poſitive or negative, as well as bc in certain caſes. For if the ſquare xx has a co-efficient, it may be freed from it by the preceding rules, and if any term be repeated more than once, their co-efficients may be added.

§. 38. To find the ſquare root or the value of x, in the equation $xx+2ax=bc$. By adding the ſquare aa of half the co-efficient $2a$ of x, we get $xx+2ax+aa=bc+aa$; the firſt ſide is a perfect ſquare, whoſe root is $a+x$, by §. 22; and therefore $x+a=\sqrt{bc+aa}$; which admits of three caſes.

I. $x=\sqrt{bc+aa}-a$; when bc is poſitive.

II. $x=$

II. $x=\sqrt{bc+aa}+a$; when the ſecond term $2cx$, is negative, and x greater than a, which is always known by the queſtion; and bc may be either poſitive or negative, but leſs than aa in the latter caſe.

III. $x=a-\sqrt{bc+aa}$; when the ſecond term $2ax$ is negative, and a greater than x, by the nature of the queſtion; but bc muſt be negative and leſs than aa.

To illuſtrate this by an example, ſuppoſe $a=6$, $bc=13$; then will $\sqrt{bc+aa}=7$, when bc is poſitive, and $\sqrt{bc+aa}=4.8$, nearly, when bc is negative.

Hence, $x=7-6$, or $x=1$, in the firſt caſe; $x=7+6$, or $x=13$ in the ſecond, when bc is poſitive, and $x=4.8+6$, or $x=10.8$, when bc is negative: laſtly, $x=6-4.8$, or $x=1.2$. in the third caſe.

This ſhort introduction will be ſufficient for ſuch readers, as want to know no more than is ſufficient in practice, or requiſite to underſtand the enſuing work: as to thoſe who are deſirous to acquire more knowledge in this ſcience, they may have recourſe to thoſe authors who have writ profeſſedly on this ſubject.

GEO-

GEOMETRY.

Definitions.

A Term or bound is the extreme of any thing.

A line has length only; the bounds of a line are points, which have no parts.

A right line lies evenly between its bounds, or is the ſhorteſt diſtance from one point to another, and a curve line lies unevenly between its bounds.

N. B. Hereafter a right line will be called a line only.

A ſurface has length and breadth; the bounds of a ſurface are lines.

A ſolid has length, breadth, and thickneſs; the bounds of a ſolid are ſurfaces.

The opening A, fig. 1. of two lines, which meet in a point, is called an angle.

Two lines AB, CD, fig. 2. which meet another line AE, ſo as to make the angles BAE, DCE at the ſame ſide equal, are ſaid to be parallel.

If a line CD, fig. 3, meets another line AB, ſo as the angles CDA, CDB, on both ſides are equal, the line CD is ſaid to be perpendicular to the line AB; and the equal angles are called right angles.

If a line ED meets another line AB obliquely, the angle ADE greater than a right angle, is called an obtuſe angle; and the angle EDB leſs than a right angle, an acute angle.

Quantity is whatever may be numbered or meaſured: And magnitude a continued quantity, ſuch as angles, lines, ſurfaces, and ſolids.

AXIOMS.

1. Quantities or magnitudes, which are equal to one and the ſame, are equal to each other.

2. If

2. If equals are added to or ſubtracted from equals, the ſums or differences are equal.

3. Magnitudes which coincide with one another, or exactly fill the ſame ſpace, are equal.

PROPOSITION, Fig. 3.

Article 1. *If a line* ED *meets another line* AB, *the angles* ADE, EDB, *at the ſame ſide of it, are together equal to two right angles.*

Let CD be perpendicular to AB; then as the greateſt angle ADE exceeds the right angle ADC, by the angle CDE, or by as much as the leaſt EDB wants of it; theſe two angles are together equal to two right angles.

PROP. Fig. 4.

2. *If two lines* AB, CD, *croſs each other in* E, *the oppoſite angles a, c, are equal.*

The angles, *a*, *b*, being together equal to two right angles by the laſt, as are alſo the angles *b*, *c*; by taking away the common angle *b*, the angle *a* will be equal to its oppoſite one *c*, by *Ax.* 2.

PROP. Fig. 5.

3. *If two parallel lines* AB, CD, *are crossed by another line* EF, *in* G, H; *the alternate angles* AGF *and* DHE, *are equal.*

The oppoſite angles *a*, *c*, are equal by the laſt, as well as the angles *b*, *c* at the ſame ſide, by *defin.* of parallels: and as both the angles *a*, *b*, are equal to the ſame angle *c*, they are equal by *Ax.* 1. For the ſame reaſon, the alternate angles AGE and FHD are equal.

Definitions.

A figure is a ſpace contained within one continued or more terms.

A plane figure lies evenly between its terms, or agrees with a right line applied to it according to any direction.

When a plane figure is bound by three lines, it is called a triangle; when by four a quadrilateral; and when bound by any number of lines whatever, a polygon. When two ſides of a triangle are equal, it is called an inſoceles; when the three ſides are equal, an equilateral; when one angle

angle is a right one, a right angle triangle; and the ſide oppoſite to the right angle, the hypothenuſe.

PROP. Fig. 6.

4. *If any ſide* AC *of a triangle* ABC *be produced, the external angle* BCD, *will be equal to the two internal oppoſite ones* A *and* B.

Let CE be parallel to the ſide AB; then the angles A, *a*, on the ſame ſide are equal, by the *Defin.* of parallels, and the alternate angles B, *b*, are equal, by *Art.* 3; the two internal oppoſite angles A, B, are then equal to the two angles *a* and *b*; that is to the external angle BCD, by *Ax.* 2.

5. Hence, the three angles of any triangle are always equal to two right angles, as being equal to the angles BCA, BCD, which are equal to two right angles, by *Art.* 1.

6. If two angles of one triangle are equal to two angles of another, the third of the one, will alſo be equal to the third of the other, as they make up two right angles.

7. When the two internal opposite angles A, B, are equal, the external angle BCD, is double either of these angles.

PROP. Fig. 7.

8. *If two triangles* ABC, DEF, *have one angle* B *in the one, equal to the angle* E *in the other, as well as the adjacent sides* AB *to* DE *and* BC *to* EF, *they will be equal in all respects; that is, the third side to the third side; and the angles opposite to the equal sides will be equal.*

Imagine the triangle DEF, placed on the triangle ABC, so that the angle E agrees with its equal B, and the side DE with its equal AB, then will the side EF, agree with its equal BC; and as the points D, F, agree with the points A, C, the side DF will likewise agree with the side AC; these triangles are then equal in all respects, by *Ax.* 3. that is, the third side DF is equal to the third side AC; the angle A to the angle D, and the angle C to the angle F opposite to the equal sides.

 9. Hence,

9. Hence, if the ſides adjacent to the equal angles B, E, are equal in the ſame triangle, the angles A, C, or D, F, are equal. For whether the ſide DE be placed on the ſide AB, or on the ſide BC, theſe triangles will equally agree in every reſpect.

PROP. Fig. 8.

10. *If the three ſides of one triangle are equal to the three ſides of another reſpectively, they are equal in all reſpects.*

Let AC be the common ſide, AB equal to AD, and CB to CD, join DB; then as AB and AD are equal, the angle ABD is equal to the angle ADB by the laſt, and as CB and CD are equal, the angle CBD is alſo equal to the angle CDB; whence the angle ABC is equal to the angle ADC, by *Ax.* 2. and ſince the adjacent ſides to theſe equal angles are equal; theſe triangles are equal in all reſpects, by *Art.* 8.

11. Hence the perpendicular drawn from the angle A between two equal ſides AB, AD, to the oppoſite ſide DB, will

bisect that side. For the angles ABD, ADB, being equal by the last, and the angles at E being right angles, the angles at A are equal, by *Art.* 6: and the triangles BAE, ADE, having one angle and the adjacent sides equal, are equal in all respects by *Art.* 8: whence the sides DE, EB, opposite to the equal angles at A, are equal,

PROP. Fig. 9.

12. *If a line* AE *makes equal angles a, b, at the same side with two parallels* AB, CD; *any other line* BE, *will also make equal angles at the same side with these parallels.*

Let E be the intersection; then the angles *a*, *b*, being equal, and the angle E common to the triangles EAB, ECD, the third EBA in the one, will be equal to the third EDC, in the other, by *Art.* 6.

Definitions.

When the opposite sides of a quadrilateral are parallel, it is called a parallelogram.

A

A Rectangle is a parallelogram, whoſe four angles are equal.

A ſquare is a parallelogram, whoſe four ſides are equal as well as the four angles.

A diagonal is a line which joins the oppoſite angular points of a quadrilateral.

The altitude of a parallelogram is the perpendicular diſtance between two oppoſite ſides, and either of theſe ſides may be the baſe.

The altitude of a triangle is the perpendicular diſtance from an angle to the oppoſite ſide, called the baſe, and the oppoſite angle, the *vertex*.

PROP. Fig. 10.

13. *The lines* AB, CD, *which join the extremities of two equal parallels* AC, BD *at the ſame ſide, are alſo equal and parallel; that is, the figure is a parallelogram.*

Draw the diagonal CB, then the alternate angles ACB, DBC, are equal by *Art.* 3: and the triangles ABC, DBC, having

having the adjacent ſides to theſe equal angles equal, are equal in all reſpects, by *Art*. 8: therefore the ſides AB, CD, oppoſite to the equal angles ACB, DBC, are equal; and ſince the alternate angles ABC, BCD, oppoſite to the equal ſides AC, BD, are equal; AB, CD, are parallel, and the figure is a parallelogram.

14. Hence, any triangle CDB, is always half the parallelogram AD of the ſame baſe and altitude.

15. As the four angles of a quadrilateral are equal to the angles of two triangles, they are equal to four right angles.

16. Since the angles of a ſquare or rectangle are equal by *Defin*; and equal to four right angles by the laſt; each of them is a right angle.

17. Parallelograms and triangles, which are between the ſame parallels, have the ſame altitude; for when AC is perpendicular to CD, its equal BD, will likewiſe be perpendicular to the ſame line.

PROP.

PROP. Fig. 11.

18. *Parallelograms* AC, EG, *or triangles, which have equal bases* AD, EH, *and the same altitude, or which is the same, are between the same parallels, are equal.*

Draw AF, DG; then as FG is equal to its opposite EH, and this equal to AD, by supposition, the figure AG is a parallelogram by the last; and as both the angles *a*, *b*, are equal to the same angle *c*, by *Defin.* of parallels, they are equal; and the adjacent sides to these equal angles, are equal as being the opposite sides of parallelograms; the triangles ABF, DCG, are equal in all respects, by *Art.* 8. By taking the triangle ABF, from the space ABGD, we shall have the parallelogram AG, and by taking the triangle DCG its equal from the same space, we get the parallelogram AC; therefore the parallelograms AC, and AG, are equal, as well as the parallelograms AG and EG; and so the parallelograms AC and EG, are equal, by *Ax.* 1.

PRO-

PROPORTIONS.

Definitions.

1. *Ratio,* is the relation between two quantities of the ſame kind, in reſpect to their contents.

2. In comparing two quantities a, b, the firſt a is called *antecedent*, and the ſecond b *conſequent*.

3. The ratio between two quantities, is the ſame as the quotient of the antecedent divided by the conſequent.

Thus the ratio of 20 to 5 is 4; that is, the antecedent contains 4 times the conſequent; and the ratio of 12 to 4 is 3, or the antecedent contains 3 times the conſequent.

4. Four quantities a, b, c, d, are ſaid to be proportional, when the ratio of the firſt a to the ſecond b, is equal to the ratio of the third c to the fourth d.

The proportion is marked thus $a : b :: c : d$, in words, a is to b as c is to d.

5. Quan-

5. Quantities are ſaid to be in a continued proportion, when the conſequent of any ratio is always equal to the antecedent of the ſucceeding ratio.

As in $a : b :: b : c :: c : d$, which ſometimes is alſo marked $:: a : b : c : d$.

6. If four quantities a, b, c, d, are in a continued proportion, the ratio of the firſt a to the third c, is ſaid to be duplicate; and the ratio of the firſt a to the fourth d, triplicate to the ratio of the firſt a to the ſecond b.

On the contrary, the ratio of the firſt a to the ſecond b, is ſaid to be ſubduplicate to that of the firſt a to the third c, and ſubtriplicate to that of the firſt a to the fourth d.

7. If there be any number of quantities, a, b, c, d, of the ſame kind, the ratio of the firſt a to the laſt d, is ſaid to be compounded of the ratios of the firſt to the ſecond, of the ſecond to the third, of the third to the fourth, and ſo on to the laſt, viz. $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{d}$.

19. *If four quantities a, b, c, d, are proportional, the product of the means is equal to that of the extremes.*

For $\frac{a}{b} = \frac{c}{d}$ by ſuppoſition; if this equality be multiplied by bd, we get $ad = bc$.

20. Hence 1°. If three quantities a, b, c, are in a continued proportion, the ſquare of the mean b is equal to the product of the extremes a, c; for $a : b :: b : c$.

2°. If four quantities a, b, c, d, are ſuch that the product bc of any two is equal to the product ad of the others, they are reciprocally proportional.

21. From hence follows theſe ſeveral changes of diſpoſition.

Directly - - -	$a : b :: c : d$.
Inverſely - - -	$b : a :: d : c$.
Alternately - -	$a : c :: b : d$.
By multiplication	$na : b :: nc : d$.
By addition -	$a : a+b :: c : c+d$.
By ſubtraction	$a : a-b :: c : c-d$.
By equality of ratios	$a+b : a-b :: c+d : c-d$.

22. If

22. If four quantities *a, b, c, d,* are proportional, their ſquares or cubes are alſo proportional.

23. If ſix lines *a, b, c, d, e, f,* are proportional, the ſquare of any antecedent *a,* is to the ſquare of its conſequent *b,* as the product of the remaining antecedents is to the product of the remaining conſequents, viz. $aa : bb :: ce : df$, ſince the product of the means is equal to that of the extremes.

24. If there be two ranks *a, b, c, d,* and *e, b, c, f,* of four proportionals, two by two, in which either the means or the extremes are equal, the four others will be reciprocally proportional. For $bc = ad$ and $ef = bc$, by *Art.* 19: and $ad = ef$, by *Ax.* 1; or $a : e :: f : d$, by *Art.* 19. N°. 2.

Hence, if there be two ranks, *a, b, c,* and *d, b, f,* of three continued proportionals, and either the means or the extremes are equal, the others are reciprocally proportional.

25. Two

25. Two ratios which are equal to a third, are equal to each other. For if $a : b :: e : f$, and $c : d :: e : f$, then $\frac{a}{b} = \frac{e}{f}$, $\frac{c}{d} = \frac{e}{f}$; and $\frac{a}{b} = \frac{c}{d}$, or $a : b :: c : d$, by *Ax.* 1.

PROP. Fig. 12.

26. *Parallelograms* AD, CD, *or triangles, which have the ſame altitude, are in the ſame proportion as their baſes* AB, BC.

For ſince parallelograms and triangles of equal baſes and equal altitudes are equal by *Art.* 18. whatever part or parts the baſe AB is of the baſe BC: the parallelograms AD and CD, may be divided into as many equal ones; and the number of thoſe in AD, will be to the number of thoſe in CD, as the baſe AB is to the baſe BC: thus if AB is to BC as 3 to 4, the parallelogram AD will contain three, and the parallelogram CD, four equal ones: this will always be the caſe, whatever ratio the baſes may have.

PROP.

On the contrary, if parallelograms or triangles have the ſame or equal baſes BD, they are as their altitudes AB, BC. This will be proved in *Art.* 48, from different principles.

PROP. Fig. 13.

27. *Equiangular triangles have their ſides, oppoſite to equal angles, proportional.*

Let the triangles ADE, ABC, have the angles at D and B equal; then will DE and BC be parallel by *Defin.* of parallels; and theſe triangles will be equiangular by *Art.* 12. draw DC and EB. The triangles ADE, DEB, having the ſame vertex E or altitude, are as their baſes, *Art.* 26. viz. ADE : DEB :: AD : DB; and the triangles ADE, DEC, having the ſame vertex D or altitude, are likewiſe as their baſes, viz. ADE : DEC :: AE : EC. As the triangles DEB, DEC, have the ſame baſe DE, and are between the ſame parallels, they are equal, by *Art.* 18. We have therefore, AD : DB :: AE : EC, by *Art.* 25, or AD : AB :: AE : AC, by *Art.* 21. N°. 5 and 6.

It may be proved in the ſame manner, that AD : AB :: DE : BC : by drawing DE parallel to AC.

PROP. Fig. 13.

28. *Equiangular triangles, are as the ſquares of their ſides oppoſite to equal angles.*

Draw AF perpendicular to BC, cutting or meeting DE in L: then the triangle ADE is to the triangle whoſe baſe and altitude are equal to DE, as AL is to DE, by *Art.* 26; and the triangle ABC, is to the triangle whoſe baſe and altitude are equal to BC, as AF is to BC, for the ſame reaſon. But AL : AF :: DE : BC, by *Art.* 27. Therefore the triangle ADE is to the triangle ABC, as the triangle whoſe baſe and altitude are equal to DE, is to the triangle whoſe baſe and altitude are equal to BC; or as the ſquare of DE is to the ſquare of BC their doubles, by equality of ratios.

PROP. Fig. 14.

29. *In a triangle* ACB, *right angled at* C, *the ſquare of the hypothenuſe* AB, *is equal to the ſquares of the other ſides* AC, CB.

Make

Make the ſquare ADEB, and thro' the right angle C, draw GL perpendicular to AB: then the triangles ACB, ALC, having the common angle A beſides a right angle, are equiangular, *Art.* 6; and AB : AC :: AC : AL, *Art.* 27; or the rectangle DL made by AB or AD and AL is equal to the ſquare of AC, *Art.* 19; it is proved in the ſame manner, from the equiangular triangles ACB, CLB, that the rectangle LE, made by LB and AB or BE is equal to the ſquare of CB: and ſince the rectangles LD, LE, are equal to the ſquare AE; the ſquare of the hypothenuſe is equal to the ſquares of the other two ſides.

CIRCLES.

Definitions. Fig. 15.

A Circle is a plane figure bound by one continued curve-line, called *circumference*, being every where equally diſtant from a point O within, called *center*.

Any line AO, drawn from the center to the circumference, is called a *radius*.

Any line AB, terminated by the circumference, is called a *chord*.

A chord divides the circle into two parts AEB, ACDB, called *ſegments*.

The chord CD, which paſſes thro' the center O is called a *diameter*, and divides the circle into two equal parts.

Any part AEB of the circumference, is called an *arc*.

Any part AOB of a circle, terminated by two radii, is called a *ſector*.

A line which touches a circle only and falls intirely without it, is called a *tangent*.

N. B. All radii are equal in the ſame circle, and alſo all diameters, being their doubles, by the definition of the circle.

Prop. Fig. 16.

30. *In the ſame circle, equal chords* AB, CD, *terminate equal arcs.*

Imagine the ſegment DEC, placed on the ſegment BFA, ſo that the chord DC, agrees

agrees with its equal BA; then will the arc DEC agree with the arc BFA; for if the equal chords DE, BF are drawn, and alſo radii drawn to their extremities, the triangles ODE, OBF, having all their ſides equal, are equal in all reſpects, *Art.* 10: ſo the point E will agree with the point F; and as this will always happen in reſpect of any two points equally diſtant from the points D, B; every point in the arc DEC, will agree with every point in the arc BFA.

31. *Hence, in the ſame circle, the angles at the center, made by radii drawn to the extremities of equal arcs, are equal.* For if the arcs AFB, CED, are equal, the chords AB, CD are equal, as well as the triangles AOB, COD, in all reſpects; and hence, the angles at O, oppoſite to the equal ſides AB, CD, are equal.

32. *The angles in the ſame circle, are proportional to the arcs they ſubtend.* For ſince equal arcs ſubtend equal angles, whatever ratio two arcs have, the angles they ſubtend will be in the ſame ratio.

N. B. Mathematicians divide the circumference of a circle into 360 equal parts, called degrees, each degree into 60 minutes, and each minute into 60 seconds. Theſe diviſions being marked on a ſemi-circle of braſs, or on an ivory rectangle, called *protractor*, ſerve to meaſure or lay down angles upon paper.

33. Hence, the ſemi-circumference or the meaſure of two right angles is 180 degrees, the quadrant or meaſure of a right angle 90 degrees, and laſtly each angle of an equilateral triangle 60, being each one third of two right angles.

PROP. Fig. 17.

34. *The angle* COD *at the center is double the angle* CAD *at the circumference, which inſiſts on the ſame arc* CD.

Draw the diameter AB, and the radii OC, OD: then as the ſides AO, OC, are equal, the triangle AOC is iſoſceles, and the external angle COB is double the internal one CAB, by *Art.* 7: and for the ſame reaſon, the external angle DOB is double

double the internal oppoſite one BAD: whence the angle COD at the center is double the angle CAD at the circumference, by *Ax.* 2.

35. As an angle at the circumference is always half the angle at the center, which inſiſts upon the ſame arc, and the angle at the center is meaſured by that arc; an angle at the circumference is meaſured by half the arc upon which it inſiſts.

Hence 1°. An angle at the circumference is leſs, equal to, or greater than a right angle, according as the arc upon which it inſiſts is leſs, equal to, or greater than a ſemi-circumference; or which is the ſame, according as the ſegment in which it is contained is greater, equal to, or leſs than a ſemi-circle.

2°. If the line AC, moves about A as a center, till it becomes a tangent to the circle at A, ſuch as AL; the angle LAC, made by a tangent, and any chord AC, will be meaſured by half the arc AC, terminated by that chord.

PROP. Fig. 18.

36. *If two lines* AB, CD, *interſect each other, within or without the circle, the parts terminated by the circumference and the point of interſection, are reciprocally proportional.*

Draw the lines AD, BC: then the triangles EAD, ECB, having the angles at D and B, which inſiſt upon the ſame arc CA equal, and the angle E common, are equiangular: and AE : CE :: ED : EB, by *Art.* 27.

If the interſection is within the circle, as at *e*, the triangles D*e*C, B*e*A, are equiangular, and D*e* : B*e* :: *e*C : *e*A.

POLYGONS.

Definitions.

A polygon is ſaid to be inſcribed in a circle, when all its angles touch, or are in the circumference; and circumſcribed, when all the ſides touch the circle.

When all the ſides of a polygon are equal, as well as its angles, it is ſaid to be regular.

PROP. Fig. 19.

37. *Equiangular Polygons, inſcribed in circles, or circumſcribed, are proportional to the ſquares of the radii.*

Draw radii to the extremities of the ſides, which will divide the polygons into as many triangles as there are ſides; and thoſe in the one will be equiangular to thoſe in the other: and the equiangular triangles AOB, COD, are as the ſquares of the ſides AO, CO, oppoſite to equal angles, *Art.* 28: and the number of triangles in the one, is equal to thoſe in the other; the polygons themſelves are, by *Art.* 21, N°. 4, proportional to the ſquares of the radii.

38. *The ſum of the ſides of equiangular inſcribed or circumſcribed polygons, are proportional to the radii.*

For the ſides of equiangular triangles, oppoſite to equal angles, are proportional, by *Art.* 27; the radius AO is to the radius CO, as the ſide AB is to the ſide CD, or

as

as all the ſides of one polygon are to all the ſides of the other.

39. *The areas of circles are proportional to the ſquares of their radii.*

As circles are the limits of all inſcribed and circumſcribed polygons, and the greateſt polygon that can be inſcribed, and the leaſt that can be circumſcribed, differ by a quantity leſs than can be conceived; and as theſe equiangular polygons, are always proportional to the ſquares of the radii; the circles their limits muſt be in the ſame proportion.

40. *The circumferences of circles are proportional to their radii.*

Since the circumference is the limit of the ſums of the ſides of inſcribed and circumſcribed polygons; and thoſe of the greateſt that can be inſcribed, and the leaſt that can be circumſcribed, differ by a leſs quantity than can be conceived; and as the ſum of the ſides of equiangular polygons, are as the radii, *Art.* 38; the circumferences their limits muſt be in the ſame proportion.

41. As

41. As the area of a polygon is equal to the product of the ſum of the ſides, and half the perpendicular to one of the ſides; ſo the area of the circle which is the limit, of the inſcribed and circumſcribed polygons, will likewiſe be equal to the product of the circumference and half the radius. The ſame thing is true in regard to any ſector.

GENERAL INVESTIGATION.

Definition.

The *general term* of a ſeries, is ſuch an expreſſion as is compoſed of one variable, and one or more conſtant quantities, that by writing 1, 2, 3, 4, &c. for the variable quantity, it becomes the firſt, ſecond, third, fourth, &c. term of that ſeries.

Thus z is the general term of the ſeries 1, 2, 3, 4, &c. of natural numbers, ſince by writing any number for z, it gives that term of the ſerios: $1+2z$ is the general

neral term of 3, 5, 7, 9. &c. for when z is 1, 2, 3, &c. you get the first, second, third term; and $2z$, is that of 2, 4, 6, 8, for the same reason.

PROP.

42. *If* z *expresses the number of terms of a series, whose sum may be expressed by the product of any number of factors, such as* z, $z+1$, $z+2$; *which exceed each other by unity, to find the general term from the sum.*

It is evident, that if the variable quantity z, be diminished by unity in the sum; that sum will be diminished by the last term, and the sum thus diminished, subtracted from the first sum, will be the general term required.

Example. If $z.z+1$ be the sum of z terms of a series; by writing $z-1$ for z, we get $z-1.z$, and this subtracted from $z.z+1$, gives $2z$ for the general term.

N. B.

N. B. The points between the factors signify multiplication: thus $z.z+1.z+2$, signifies $z \times \overline{z+1} \times \overline{z+2}$.

If $z.z+1.z+2$, be the sum, by writing $z-1$ for z, we get $z-1.z.z+1$, and this sum subtracted from the first, gives $3z.z+1$; if $z.z+1.z+2.z+3$ be the sum, by writing $z-1$ for z, we get $z-1.z.z+1.z+2$; and this subtracted from the former gives $4z.z+1.z+2$, for the general term. If the number of factors be what it will, the general term will always be found in the same manner.

Hence to find the general term of a series from the sum, observe the following.

I. General Rule.

43. *Multiply the sum by the number of factors, and strike out the last factor.*

N. B. Whether the sum is multiplied by a constant quantity, or the factors increase or decrease, the rule is the same; provided the value of z be increased when the factors decrease, instead of being diminished, and the first sum be subtracted fom the second.

For

For $az \cdot \overline{z-1}$, gives $2az$; $az \cdot \overline{z-1} \cdot \overline{z-2}$, gives $3az \cdot \overline{z-1}$; and $az \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3}$, gives $4az \cdot \overline{z-1} \cdot \overline{z-2}$.

On the contrary, ſince the general term of a ſeries is found by multiplying the ſum of the factors by the number of them, and ſtriking out the laſt factor; to find the ſum of a ſeries from its general term, obſerve the following.

II. General Rule.

44. *Increaſe the factors by one more factor, and divide by the number of factors thus increaſed.*

Thus the general term az, gives $\frac{1}{2}az \cdot \overline{z+1}$, for the ſum; $az \cdot \overline{z+1}$ gives $\frac{1}{3}az \cdot \overline{z+1} \cdot \overline{z+2}$; the general term $a+z$, gives $az+\frac{1}{2}z \cdot \overline{z+1}$; and $a+nz$, gives $az+\frac{1}{2}nz \cdot \overline{z+1}$. Whether the general term conſiſts of one or more parts, the rule will be the ſame for each part, and their ſums added together, the whole ſum.

N. B. When the firſt value of the variable quantity z is unity, the factors of the

the ſum will be of an increaſing progreſſion, and when that value is o, of a decreaſing one. Thus in the general term $a+nz$, the firſt value of z will be unity, when the firſt term of the ſeries is $a+n$, but o when the firſt term is a.

PROP.

45. When the common difference between the factors vaniſhes, or becomes o, the quantities become continued, ſuch as lines, ſurfaces, and ſolids: and the two preceding general rules remain the ſame.

For the ſum $z.\overline{z+1}$, becomes zz, and its general term $2z$ remains the ſame: the ſum $z.\overline{z+1}.\overline{z+2}$, becomes z^3, and its general term $3z.\overline{z+1}$, becomes $3zz$: and in general the ſum z^n gives nz^{n-1}, for its general term.

Though the tranſition from diſcontinued to continued quantities is very natural, yet as ſome readers may not perhaps readily perceive the connection, we ſhall endeavour to explain it by the Fig. 20.

Divide the line AE into any number of equal parts; at the points of diviſion erect

the perpendiculars B*k*, C*l*, D*m*, E*n*, to AE; and let these perpendiculars express the terms of the series: Then the difference between the sums of all the terms, except the last, and the sum of all the terms gives the last term E*n*, whatever the quantities are, whose ratios are expressed by an arithmetical progression.

Now, if the terms are superficial, each term represents a certain number of superficial units, and when their interval vanishes, the area A*n*E, will express their sum, and when the terms are expressed by solid units, the sum of all the terms will be found as usual: but when the common difference between the factors vanishes, the sum will express one continued solid. Thus for example; if the terms represent the different horizontal ranges of a pile of shot, the sum will give the number of shot of that pile; and when the common difference between the factors vanishes, the sum will give the content of the solid of that figure.

Otherwise

Otherwise.

Suppose the surface A*n*E, described by the motion of a line AT perpendicular to AE, then the difference D*mn*E between the spaces A*n*E, A*m*D, divided by the difference DE between the bases, will give a quotient less than E*n*, and greater than D*m*. For the rectangle of DE and E*n* will be greater, and the rectangle of DE and D*m* less, than the space D*n*, or the rectangle of the quotient and DE: and these rectangles having the same base DE, are as their altitudes: when DE vanishes, the quotient which is less than E*n*, and greater than D*m*, will become equal to E*n*, or to the general term required.

46. Hence, it may be proved in the same manner, that the general term of the solid described by the area A*n*E about the axis AE, or about AT the perpendicular to AE, is the surface described by E*n* in that revolution: Since the solid described by the difference D*mn*E, divided by DE gives a surface, which, when DE

vanishes, becomes that described by En for the general term.

By an example we shall shew, that the latter way of finding the general term, agrees exactly with the former. Let AD$=x$, AE$=z$, and z^3 express the area or solid AnE: then by the same reason, x^3 expresses the area or solid AmD, and the difference z^3-x^3, divided by the difference $z-x$, gives $zz+zx+xx$; which, when the difference $z-x$ vanishes, becomes $3zz$ for the general term of the sum z^3, which is the same as before.

N. B. It is to be observed, that the general term of a series, must always be expressed by the variable base and constant quantities, free from surds or irrational quantities; since the foregoing rules depend upon this supposition.

PROP.

47. Let the series be the arithmetical progression a, $a+n$, $a+2n$, $a+3n$, &c. whose general term is $a+nz$, and 0, 1, 2, 3, the values of z, which beginning

ginning with o, ſhews that the factors muſt decreaſe, by the obſervation after *Art.* 44. Hence by the general rule, *Art.* 44. we get $az+\frac{1}{2}nz.\overline{z-1}$ for the ſum of z terms.

Example. If the ſeries be the natural numbers 1, 2, 3, 4, 5, then will $a=n=1$, and the ſum becomes $\frac{1}{2}z.\overline{z+1}$: or if z is 29, then 29×15 or 435 is the ſum of 29 terms.

If the ſeries be the odd numbers 5, 7, 9; then $a=5$, $n=2$, and $5z+z.\overline{z-1}$ is the ſum; or $z.\overline{z+4}$, when reduced. If $z=20$, then 20×24, or 480, is the ſum of 20 terms.

If the ſeries be the even numbers 2, 4, 6, 8; then $a=n=2$, and $2z+z.\overline{z-1}$ is the ſum; or $z.\overline{z+1}$, when reduced, and if $z=20$, then 20×21, or 420, is the ſum of 20 terms.

PROP. Fig. 21.

48. When the difference between the factors vaniſhes, the ſum $az+\frac{1}{2}nz.\overline{z-1}$, becomes $az+\frac{1}{2}nzz$, which expreſſes the

area of a Trapezium ABCD, whose opposite sides AB, DC are parallel, and $AB=a$, $EF=z$, $DC=a+nz$, or the general term. Hence,

1°. If a be made o, the sum becomes $\frac{1}{2}nzz$, which expresses the area of the triangle LCD, whose altitude LF is z and the base CD, nz, or the general term.

2°. When n is made o; then the sum becomes az; which expresses the area of a rectangle or parallelogram ABGH, whose base AB is a, and altitude EF is z.

N. B. It must be observed in general, that n expresses always the ratio of the base to that part of the perpendicular, which is to be added to the part a, in order to make up the whole perpendicular, or the general term.

PROP.

49. Let the series be the squares of the terms of the following arithmetical progression a, $a+n$, $a+2n$, $a+3n$, &c. whose general term is $aa+2anz+nnzz$, the square of $a+nz$; and o, 1, 2, 3, the values

values of z. Now the sum of the two first terms $aa + 2naz$, is $aaz + anz . z - 1$, by *Art.* 44; and as $zz = z + z . z - 1$, whose sum by the same *Art.* is $\frac{1}{2}z . z - 1 + \frac{1}{3}z . z - 1 . z - 2$, or $\frac{1}{6}z . z - 1 . 2z - 1$, when reduced under the same denomination; this last sum being multiplied by nn, and added to that of the two first terms, gives the sum required, viz. $aaz + naz . z - 1 + \frac{1}{6}nnz . z - 1 . 2z - 1$.

1°. Hence, when $n = 1$, the sum becomes $aaz + az . z - 1 + \frac{1}{6}z . z - 1 . 2z - 1$; which is the number of shot contained in an unfinished square pile.

2°. The trapezium of shot, whose upper base is a, and the corner row z, is by *Art.* 47, $az + \frac{1}{2}z . z - 1$; and if m denotes the number of trapeziums, then their sum will be $maz + \frac{1}{2}mz . z - 1$; which being added to the sum of the unfinished square pile, gives the sum of an unfinished rectangular pile, whose sides of the upper base are a and $a + m$.

General Rule for unfinished Piles.

To twice the length and breadth of the upper base, add the corner row less one; to the product of these two numbers add one third of the product of the corner row less one, by the corner row more one;

And multiply the sum by one fourth part of the corner row.

For the product of $2a+z-1$ by $2a+2m+z-1$, added to the product $\frac{1}{3}z-1 . z+1$, and multiplied by $\frac{1}{4}z$, gives the sum proposed.

1°. Hence, when the rectangular pile is compleat; then a becomes unity, and if $a+m=b$, the sum becomes $3b+2.z-1 \times$ by $\frac{1}{6}z . z+1$.

2°. When b, is unity, then $\frac{1}{6}z . z+1 . 2z+1$, will be the sum of a compleat square pile.

3°. When $m=0$, and the sum found by the general rule, is divided by 2; it will become the sum of an unfinished triangular pile; and $\frac{1}{6}z . z+1 . z+2$, will be the sum, when the pile is compleat.

These three cases might have been investigated separately; the second from the series of rectangles ab, $a+1 . b+1$, $a+2 . b+2$, &c; and the third from that of triangles $\frac{1}{2}a . a+1$, $\frac{1}{2}a+1 . a+2$, &c; but as this has been done already in our artillery; it will be needless to explain them any further.

Definitions.

A *pyramid* is a solid whose base is any plane figure, and terminated above in a point, as marked by the letter A.

When the base of a pyramid is a circle, it is called a *cone*; as marked B.

A *prism* is a solid whose opposite bases are two equal and parallel plane figures, as marked C.

When the opposite bases of a prism are circles, it is called a *cylinder*, as marked D.

A *sphere* is a solid, described by a semicircle, about its diameter as an axis.

Any part of a sphere terminated by a circle perpendicular to its axis, a *segment*.

The upper point of a pyramid or cone is called the *vertex*.

The perpendicular distance between the vertex of a pyramid or cone, and the base, or between the upper and lower bases of a prism or cylinder, is called the *altitude*.

PROP. Fig. 21.

50. When the difference between the factors vanishes, the sum found in *Art*. 49. becomes $aaz+anzz+\frac{1}{3}nnz^3$; which is the content of an unfinished square pyramid, whose upper base AB is aa, altitude EF is z, and lower base CD is the square of $a+nz$, or the general term.

If we make $b=a+nz$; then will $bb=aa+2anz+nnzz$, and $ab=aa+anz$; hence $\overline{aa+ab+bb}\times$ by $\frac{1}{3}z$, will be the content of this solid. But circles, and all equiangular inscribed figures in circles, are as the squares of their radii; all unfinished pyramids or cones, are equal *to the sum of the opposite planes added to a mean geometrical plane between them, and the sum multiplied by one third of their altitudes*.

 1°. When

1°. When $a=0$, then will $\frac{1}{3}bbz$ be the ſum, which is the content of a compleat pyramid or cone LDC, *equal to the product of its baſe bb multiplied by one third of its altitude z.*

2°. When $n=0$; the ſum then becomes aaz, which is the content of a priſm or cylinder ABGH, *equal to the product of its baſe and altitude.*

PROP. Fig. 22.

51. *To find the content of a ſemi-ſphere, deſcribed by the quadrant* OBA *about the axis* OA.

Draw PM perpendicular to OA, make the radius $OA=a$, $OP=x$, and $PM=y$; then the right angled triangle POM gives $yy=aa-xx$, by *Art.* 29; and if the ratio of the radius to its ſemi-circumference is as unity to r; then unity is to r as the radius PM (y) is to ry its ſemi-circumference, by *Art.* 40; and as the area of a circle is equal to the product of the radius by half its circumference, by *Art.* 41; we ſhall have $y \times ry$ or ryy, for the circle

deſcribed

described by the radius PM. This area or its equal $raa - rxx$, being the general term, by *Art.* 46. of the solid described by the segment OBMP of the circle, that solid will be $raax - \frac{1}{3}rx^3$, by *Art.* 44; and when OP becomes OA or $x = a$, we get $\frac{2}{3}ra^3$, for the content of the semi-sphere.

52. Hence, as the content of a cylinder, whose base is raa, and altitude a, is ra^3, by *Art.* 50, N°. 2, and that of a cone of the same base and altitude as those of the cylinder, is $\frac{1}{3}ra^3$, by the same article, N°. 1: The circumscribed cylinder, semi-sphere, and inscribed cone, are to each other, as the numbers 3, 2, 1.

53. Since the content of the cone described by the triangle OPM is $\frac{1}{3}rxyy$, or its equal $\frac{1}{3}raax - \frac{1}{3}rx^3$; the difference between the solid $raax - \frac{1}{3}rx^3$, described by the segment OBMP and this cone, gives $\frac{2}{3}raax$ for the content of the solid, described by the sector BOM, about the axis OA.

54. If the radius OA (a) be supposed variable; then the surface described by the

the arc BM, will be the general term of the ſolid deſcribed by the ſector BOM, by *Art.* 46; and ſince all the triangles OPM, made by the perpendiculars drawn from the extremities of the radii OM to the baſe OA, are equiangular; OP will be in a conſtant ratio to OM; let this ratio be $na = x$: then the ſolid $\frac{2}{3}raax$ will be equal to $\frac{2}{3}rna^3$; and as a is variable, gives $2rnaa$ for the ſurface deſcribed by the arc BM, by *Art.* 43; or becauſe $na = x$, that ſurface becomes $2rax$: and when $x = a$, we get $2raa$ for the ſurface of the ſemi-ſphere.

55. Since the content of a cylinder whoſe baſe is raa and altitude b, is $raab$, by *Art.* 50. N°. 2: when a is variable, we get $2rab$, by *Art.* 43, for the general term, or the ſurface of that cylinder, by *Art.* 46; which when b becomes a, will be $2raa$, equal to that of the ſemi-ſphere; and when b becomes x, it will then become $2rax$, equal to that deſcribed by the arc BM, correſponding to the altitude OP, of the cylinder.

PROP.

PROP. Fig. 22.

56. *To find the content of the circular segment* OBMP.

Let the radius OA be unity, and OP$=x$; then will PM$=\sqrt{1-xx}$, be the general term of that area; which not being expressed according to the foregoing rules, the square root must be extracted by the common rule of Algebra, which gives PM$=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-$&c. for the general term: whose sum or area, by *Art.* 44. is, $x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-$ &c.

As this series converges but slowly unless x is very small, we must find the area of the sector BOM. Since the area of the triangle OPM is equal to half the product of its base OP, and altitude PM, or to $\frac{x}{2}-\frac{x^3}{4}-\frac{x^5}{16}-\frac{x^7}{32}-\frac{5x^9}{256}$; this being subtracted from the area of the segment, gives $\frac{x}{2}+\frac{x^3}{12}+\frac{3x^5}{80}+\frac{5x^7}{224}+\frac{35x^9}{2302}$, for the area of the sector: but if the arc

BM

BM be called z; then will $\frac{1}{2}z$ be also equal to the same sector: consequently the equality of these two values gives $z = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} +$, &c.

If $a = x$, $b = \frac{1}{2}axx$, $c = \frac{3}{4}bxx$, $d = \frac{5}{6}cxx$, $e = \frac{7}{8}dxx$, &c. then will $z = a + \frac{1}{3}b + \frac{1}{5}c + \frac{1}{7}d + \frac{1}{9}e$.

Hence, if the arc BM be 30 degrees, its sine x will be equal to half the chord of 60, that is, $x = \frac{1}{2}$; and hence $a = \frac{1}{2}$, $b = \frac{a}{8}$, $c = \frac{3b}{16}$, $d = \frac{5c}{24}$, $e = \frac{7d}{32}$, $f = \frac{9e}{40}$, $g = \frac{11f}{48}$, $h = \frac{13g}{56}$, $i = \frac{15h}{64}$; and these values reduced into decimals, give,

a) .5	and	.50000,000
b) .0625		2083,334
c) .01171,875		234,375
d) .00244,141		34,877
e) .00053,406		5,934
f) .00012,016		1,093
g) .00002,754		212
h) .00000,639		43
i) .00000,149		9
		.52359,877

Now

Now the ſum .52359,877, being the ſixth part of half the circumference; which therefore multiplied by 6, gives 3.14159,262, for half the circumference, whoſe radius is unity, true in all its places, excepting the laſt 2, which ſhould be 5. Hence, the radius is to half its circumference, or the diameter to the whole circumference, as unity is to 3.14159265.

If the diameter of a circle be 7; the circumference will be 21.99, or 22 nearly, which is the proportion of *Archimedes*. But if the diameter be 113, the circumference will be 354.9999 or 355 exceedingly near, which is that of *Snellius*. Having the circumference expreſſed by parts of its diameter, any arc may be expreſſed in the ſame manner, when the number of degrees it contains are known, by ſaying 180 degrees is to the number of degrees of the arc, as 3.14159. &c. is to the parts required. The arc being expreſſed in parts of the diameter, the ſector and ſegment may likewiſe be found.

GREATEST

Greatest and Least Quantities.

If an expression, composed of one variable quantity and its powers, with constant co-efficients, be such; that when the variable quantity increases, the value of the expression increases for a while, and then decreases, it has a value greater than the rest; or if the value decreases for a while, and increases afterwards, while the variable quantity increases, it has a value less than any other; which values we are to find.

Prop.

57. *To find the greatest and least quantities of a given expression.*

Since the expression contains a greater or least value by supposition, it must have values in its increase, before it becomes the greatest, which are equal to those in its decrease after the greatest: or if the expression contains a least value, it must have values in its decrease, before it becomes

comes the least, that are equal to those in its increase after the least.

If, therefore, we suppose two different values of the variable quantity, which produce the same or equal values of the expression; then their difference will be equal to o, and when divided by the difference between the two values of the variable quantity, it will still remain equal to o: and when the two values of the variable quantity are made equal, we shall have that which makes the expression the greatest or least required.

But we have shewn, after *Art*. 46, that the difference between two expressions, being divided by the difference between the two variable quantities, and supposing them equal, the quotient becomes the general term. Therefore, *the general term of an expression being made equal to* o, *will determine the value of the variable quantity, which makes the expression the greatest or least*.

58. Hence, if the variable quantity has but one value, the expression has but one

one greateſt or leaſt; and if it has more than one, the expreſſion has greateſt and leaſt values alternately; but ſometimes the leaſt between two greateſt is o or negative, and the greateſt between two leaſt is infinite.

N. B. When the greateſt or leaſt is found, to know which it is, take any two values for the variable quantity, one greater and the other leſs than that found; then if both values of the expreſſion thus found are leſs, the quantity will be the greateſt, and if both are greater it will be the leaſt.

Examples.

59. *To divide a given line $2a$ into two parts, ſo that their product ſhall be the greateſt.*

Let x be one of the parts, then will $2a-x$ be the other, and $2ax-xx$ the product, which is to be the greateſt; whence by the rule in *Art.* 57. its general term muſt be equal to o: which gives $2a-2x=0$, or $x=a$, by *Art.* 43.

 Hence,

Hence, when the given line is biſected, the product will be the greateſt.

When x is made equal to $2a$ or o, the expreſſion $2ax-xx$ becomes o; which ſhews that the quantity found is the greateſt.

60. *To divide a given line 2a equally and unequally, ſo that the product of the unequal parts multiplied by the middle part ſhall be the greateſt.*

Let x be the middle part, then will $a+x$ and $a-x$ be the unequal ones, and $aax-x^3$ the product, which is to be the greateſt; and hence by *Art.* 57 and 43, we get $aa-3xx=0$, whoſe roots are $x=a\sqrt{\frac{1}{3}}$, and $x=-a\sqrt{\frac{1}{3}}$.

Now when x is leſs than a, the product $aax-x^3$, is the greateſt; becauſe if x is equal to a or x, the expreſſion becomes o.

But when x is greater than a, the root $x=-a\sqrt{\frac{1}{3}}$, ſhews that the expreſſion can have no leaſt, and therefore the greateſt we have found is the only one the expreſſion can have.

61. *To divide a given line a into two parts, so that the square of one multiplied by the other shall be the greatest.*

Let x be one of the parts, then will $a-x$ be the other, and $axx-x^3$ the product, which is to be the greatest: hence we get $2ax-3xx=0$, by *Art.* 43, or $2a=3x$: which shews that if the line be divided into three equal parts, the product of the square of two thirds multiplied by one third, will be the greatest. For when x is equal to a or 0, the expression becomes 0.

Thus, for instance, if a number 12 be divided into three equal parts; each will be 4; and the square of 8 multiplied by 4, gives 256 for the greatest product.

Most of the mechanical problems that I have met with, are solved by the preceding examples, as may be seen in my works; and as they are sufficient to explain this extensive method, we shall say no more of it.

LOGARITHMS.

62. *To find the ratio of any two given numbers n and m.*

By the definition of compound ratios, the ratio of n to m, is compounded of all the intermediate ratios that poſſibly can be between theſe two numbers; and the ratio of any two numbers is equal to the quotient of the antecedent divided by the conſequent. Hence if $1+x$ be any number between n and m; then the ratio of n to $1+x$, will be expreſſed by $\frac{n}{1+x}$; which therefore is the general term of the ſum of all theſe ratios.

Now by a continual diviſion of n by $1+x$, we get $n\times$ by $1-x-xx-x^3-x^4-$ &c. whoſe ſum by *Art.* 44. is $n\times$ by $x-\frac{1}{2}xx-\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5-$, &c.

If $\frac{n}{1-x}$ be the ratio, then by the ſame manner of proceeding as before, the ſum will be $n\times$ by $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5$, &c.

Becauſe

Becauſe the ratios of a rank of geometrical proportionals, ſuch as 1, z, z^2, z^3, z^4, &c. are in an arithmetical progreſſion, they are called the logarithms of theſe proportionals; whoſe properties are, that inſtead of multiplying and dividing any two numbers, the ſum or difference of their logarithms, will be the logarithm of the product or quotient; and inſtead of ſquaring and cubing any number, twice or thrice its logarithm will be the logarithm of its ſquare or cube; likewiſe one half or one third of the logarithm of any number, will be that of its ſquare o cube root.

63. Hence, if the logarithm of $1-x$ be ſubtracted from the logarithm of $1+x$, we get the logarithm of $\frac{1+x}{1-x}$; but the logarithm of $1-x$ being leſs than unity, is negative, its poſitive value muſt be added; which gives $2n\times$ by $x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\frac{1}{9}x^9+$, &c: or if $a=2nx$, $b=axx$, $c=bxx$, $d=cxx$, &c, that logarithm will be $a+\frac{1}{3}b+\frac{1}{5}c+\frac{1}{7}d+$ &c.

As the number n may be more or leſs, there may be various logarithms; when n is unity, the logarithms are called natural or hyperbolic: but as it has been found convenient, that the logarithm of 10 ſhould be unity; the number n muſt be found in the following manner.

Let p be the natural logarithm of $\frac{5}{4}$ and q that of $\frac{128}{125}$: then will $10p+3q$ be the natural logarithm of 10. For if x be the logarithm of 2, and y that of 5; then as $\frac{5}{4}=\frac{5}{2\times 2}$, twice the logarithm x of 2, ſubtracted from y that of 5, will be equal to p that of $\frac{5}{4}$, or $y-2x=p$; and ſince $\frac{128}{125}=\frac{2^7}{5^3}$, three times the logarithm y of 5, ſubtracted from 7 times x that of 2, will be equal to the logarithm q of $\frac{128}{125}$, or $7x-3y=q$. Now 10 times $y-2x=p$ gives $10y-20x=10p$, and 3 times $7x-3y=q$, gives $21x-9y=3q$; and theſe two laſt equalities added gives $x+y=10p+3q$.

64. *To find the natural logarithm of* $\frac{5}{4}=\frac{1+x}{1-x}$.

By

By crofs multiplication we get, $5-5x=4+4x$, or $x=\frac{1}{9}$; and this value wrote into thofe of a, b, c, d, gives $a=\frac{2}{9}$, $b=\frac{a}{81}$, $c=\frac{b}{81}$, $d=\frac{c}{81}$, &c. which being reduced into decimals give,

.22222,22222 (*a*)	and	.22222,22222
274,34843 (*b*)		91,44947
3,38702 (*c*)		67741
4181 (*d*)		597
52 (*e*)		6
		.22314,35513 $=p$
		2.23143,5513 $=10p$.

65. *To find the natural logarithm of* $\frac{128}{125}=\frac{1+x}{1-x}$.

By reducing this equality we get $x=\frac{3}{253}$; and hence $a=\frac{6}{253}$; by this means the values of b and c are found, and being reduced into decimals give

.02371,55150	and	.02371,54150
33345		11115
5		1
		.02371,65266 $=q$.
		.07114,95798 $=3q$.

hence $10p+3q=2.30258,50930=$ *log.* 10.

 As

As this logarithm multiplied by n is equal to unity the tabular logarithm of 10; by dividing unity by this logarithm, we get $n = 0.43429,4482$.

Hence, if the natural logarithm of any number be multiplied by the number n, the product will be the tabular logarithm of that number, and on the contrary, if the tabular logarithm of any number be multiplied by the natural logarithm 2.30258,5093, the product will be the natural logarithm of that number: this last rule is very useful in the higher parts of mathematics.

66. *To find the tabular logarithm of* 9.

Suppose $\frac{10}{9} = \frac{1+x}{1-x}$; then will $x = \frac{1}{19}$, $a = \frac{2n}{19}$, $b = axx$, $c = bxx$, $d = cxx$; by writing these values into those of a, b, c, d, when reduced into decimals we get

.04571,52086	and	.04571,52086
12,66349		4,22116
3508		702
10		2
		.04575,74906 = log. $\frac{10}{9}$.

As

As the logarithm of 10 is unity, and 10 divided by $\frac{10}{9}$ gives 9; the logarithm just found, subtracted from unity, gives 0.95424,25094 for the logarithm of 9; and half the logarithm of 9, gives 0.47712,12547 for the logarithm of 3, the square root of 9.

67. *To find the tabular logarithm of* 8.

Suppose $\frac{81}{80} = \frac{1+x}{1-x}$; then will $x = \frac{1}{161}$,

$a = \frac{2n}{161}$, $b = axx$, and
$$\begin{array}{r} .00539,49625 \\ 693. \\ \hline .00539,50318 \end{array}$$

This logarithm subtracted from twice the logarithm of 9, gives 1.90308,9987 for the logarithm of 80; and unity subtracted from this last found, gives .90308,9987 for the logarithm of 8.

One third of the logarithm of 8, gives .30102,99956, for the logarithm of 2 the cube root of 8; twice the logarithm of 2, gives .45154,49935, for that of 4, and the sum of the logarithms of 2 and 3, gives .77815,12504 for that of 6.

68. *To*

68. *To find the tabular logarithm of* 7.

Suppose $\frac{49}{48} = \frac{1+x}{1-x}$, then will $x = \frac{1}{97}$, $a = \frac{2n}{97}$, and $b = axx$; and these values reduced into decimals gives

$$\begin{array}{r} .00895,45254 \\ 3172. \\ \hline .00895,48426 \end{array}$$

Now because $6 \times 8 = 48$; the sum of the logarithms of 6 and 8 taken from the logarithm here found, gives 1.69019,6080 for the logarithm of 49; and half this gives 0.84509,804 for that of 7, the square root of 49.

In the same manner may be computed, all the logarithms of the prime numbers, when those of the even ones next below and above are known. For in any three numbers which exceed each other by unity, the square of the middle one exceeds the product of the other two by unity, and their difference divided by their sum, will be always equal to the value of x. Thus to find the logarithm of the next prime number 11, whose square 121 divided by the product 120 of

of 10 and 12, gives $\frac{121}{120}$; whoſe logarithm is found by $\frac{121}{120}=\frac{1+x}{1-x}$, when $x=\frac{1}{241}$; to which if the ſum of the logarithms of 10 and 12 are added, then half the ſum will be the logarithm of 11. Again the logarithm of the next prime number 13, is found by dividing its ſquare 169, by the product 168 of 12 and 14; and finding the logarithm of the quotient, when $x=\frac{1}{337}$, proceed as before; and ſo on in regard to all the reſt: It may be obſerved that the logarithms of large numbers are computed by fewer ſteps; and when they exceed 500, by addition and ſubtraction only.

TRIGONOMETRY,

Is the ſcience by which the ſides and angles of a triangle are computed, when any three are given, of which there muſt be one ſide at leaſt. To do this it is neceſſary to divide the circumference into degrees, minutes, and ſeconds, as well as

to

to compute certain lines drawn without and within the circle.

Definitions. Fig. 23.

The perpendicular PM to the radius CA, is called the *ſine*, of the arc AM; as likewiſe that of its ſupplement to half the circumference.

The part CP of the radius between the center and the ſine, is called *coſine* of the arc AM; and the remainder PA of the radius, the *verſed ſine*.

The perpendicular AT to the radius AC, terminated by the radius CM produced, is called the *tangent*; and the line CT, the *ſecant* of the arc AM.

If the radius CB be perpendicular and MQ, B*t*, parallel to the radius CA; then QM is the ſine, CQ the coſine, and B*t* the tangent of the arc BM, which is the complement to AM of a quadrant.

Note, That the ſine and coſine of an arc AM, are the ſame as the coſine and ſine of its complement BM to that arc.

PROP.

PROP.

69. *Any arc* AM, *being expressed by parts of the radius; to find its sine and cosine.*

If PM$=x$, and z the arc AM; then $z=x+\frac{1}{6}x^3+\frac{3}{40}x^5+$, &c. by *Art.* 56, when the radius is unity; by cubing each side of this equation and neglecting the higher powers of x, we get $z^3=x^3+\frac{1}{2}x^5$, and $z^5=x^5$: now the first equality being divided by 6, and the second by 120; then when the values of x and its powers are wrote into that of z, we get $x=z-\frac{1}{6}z^3+\frac{1}{120}z^5$ nearly, if z be but small.

Having the sine PM of an arc, its cosine CP, is found by the right angled triangle CPM: the tangent AT and secant CT as well as the cotangent, are found by proportion.

Example.

The radius is to half the circumference as unity is to 3.14159, &c. the arc of a minute or one 60th part of a degree will be found, if half the circumference,

3.14159, &c. be divided by 60 times 180, or 10800; which gives .00029,0888, for that arc or its ſine nearly: and if we ſubtract the ſquare of this ſine from the ſquare of the radius or unity; then the ſquare root of the difference gives .99999,996, for the coſine of an arc of one minute.

Prop. Fig. 24.

70. *If there be two arcs* AN, AL, *the laſt not exceeding a quadrant; it is required to find the relations between their ſines and coſines, and of half their difference* LM.

Let DL, PM, QN, be the ſines of thoſe arcs reſpectively, draw the chord LN, and the radius CM perpendicular to LN; from the interſection R, and the point N, draw likewiſe RE, NF, parallel to CA meeting DL in E, F: then as LR=RN, we have LE=EF and FN=DQ. If the radius CM be unity, and the ſine PM=s: The ſimilar triangles CMP and LFN, give 1 : s : : 2LR : FN; or DQ=2s×LR; by adding CD to both ſides, then will CQ=

$CQ = CD + 2s \times LR$: and since CD and ER are parallel, we have $1 : s :: CR : DE = s \times CR$ by *Art.* 27. But DL, DE, DF or QN, are in an arithmetical proportion; that is, $2DE = DL + QN$; or because $DE = s \times CR$, we get $QN = 2s \times CR - DL$, by transposition.

71. Hence, if LD passes thro' the center C, then will FN be the sine of the arc LN double of LM; and the triangles CRL, LFN, will be similar; hence CL (1) CR :: LN, or $2LR : FN = 2LR \times CR$. Which shews, *that twice the product of the sine and cosine of any arc, is equal to the sine of double of that arc.*

72. If the arc LM be one minute, its sine LR is .00029,0888, and cosine CR, .99999,996, by the example above; these values being wrote into the values of CQ and QN, give $CQ = CD + 2s \times .0029,0888$, and $QN = 2s \times .99999,996 - LD$. By means of these equations, the sines and cosines of any arc, from one minute to 30 degrees, are found by one multiplication only.

Example.

Example.

73. Let LD be the radius, then will the ſine s or PM, be that of 89 degrees and 59 minutes; that is, $s = .99999,986$, by *Art.* 69; and by writing this value, and unity for DL, we get $QN = .99999,9866$, and $CQ = .00058,1876$, for the ſine and coſine of an arc of 89 degrees and 58 minutes.

The next ſine and coſine are found by ſuppoſing DL the ſine of 89 degrees and 59 minutes; and the ſine PM (s) that of 89 degrees, 58 minutes. The operations may be thus continued to an arc of 30 degrees.

74. If the arc AM be 30 degrees, then its ſine PM (s) will be equal to $\frac{1}{2}$, and the equations in *Art.* 70, become in this caſe $QN = CR - DL$, and $CQ = CD + LR$; which ſhews that the ſines and coſines of all arcs leſs than 30 degrees, may be found by addition and ſubtraction only.

PROP.

Prop. Fig. 25.

75. If from one of the acute angles C, in a right angled triangle CDE as center, an arc AM be deſcribed with any radius CA, and the tangent AT as well as the ſine PM be drawn; the right angled equiangular triangles give,

$$\left\{\begin{array}{l} CP : PM :: CA : AT :: CD : DE \\ CP : CM :: CA : CT :: CD : CE \end{array}\right\}$$

That is, *the coſine is to the ſine*, or *the radius is to the tangent, as the ſide next to the acute angle is to the oppoſite ſide.*

And, *the coſine is to the radius*, or *the radius is to the ſecant, as the ſide next to the acute angle is to the hypothenuſe.*

Caſe I.

76. *The ſides* CD, DE, *next to the right angle being given, to find the angle* C.

Let CD = 100, and DE = 60; then CD (100) is to DE (60) as the radius 100000, is to the tangent 60000 of the angle C; which is found by the tables to

be 30d. 58m. nearly; and its complement to a right angle gives 59d. 2m. for the angle E.

Case II.

77. *The side* CD *and the acute angle* C *being given; to find the other side* DE.

Let CD=100, and the angle C 40 degrees: then the radius 100000 is to the tangent 83909 of 40d. as the side CD (100) is to the side DE=83.9.

Case III.

78. *The side* CD, *and the angle* C *being given; to find the hypothenuse* CE.

Let CD=100, and the angle C 40 degrees: then the radius 100000, is to the secant 130540 of 40d. as CD (100) is to the hypothenuse CE=130.54.

Prop. Fig. 26.

79. *In any unequal sided triangle* ABC, *where a perpendicular* CD *is drawn to the base* AB; *the base* AB, *is to the sum of the sides* AC, BC; *as their difference is to the sum or difference between the segments* BD,

AD *of the base, according as the angle* A *is obtuse or acute.*

From the angle C as center describe an arc thro' the point A, cutting the base in G, the side BC in F, and when produced in E; then by the property of the circle, *Art.* 36, we have $CE=CA$, $DA=DG$, and $BA : BE :: BF : BG$, or $BA : BC+CA :: BC-CA : BD\mp DA$.

PROP. Fig. 27.

80. *In any unequal sided triangle* ABC, *the sum of the sides* BC, AC, *is to their difference, as the tangent of half the sum of the opposite angles* A, B, *is to the tangent of half their difference.*

Let the same construction be as the last, draw AF, AE, and FG parallel to AE; then the external angle ACE, is equal to the two internal opposite ones, A, B, by *Art.* 4, and double the angle CFA at the circumference, by *Art.* 34; and because CE and CA are equal, the angle CFA or CAF, will be equal to half the sum of the angles A, B, and the angle

FAG half their difference; becauſe when it is added to half the ſum, it gives the greateſt angle A, and when ſubſtracted from half the ſum CFA, gives the leaſt B, by *Art.* 4.

Now becauſe the angle FAE, is contained in a ſemi-circle, it will be a right angle, by *Art.* 35, N°. 1; AE will be the tangent of half the ſum, and FG the tangent of half the difference of thoſe angles, when FA is the radius: hence the parallels FG and EA give (BE : BF ::) BC + CA : BC − CA :: EA : FG.

Prop. Fig. 28.

81. *In any triangle* ABC, *the ſides are proportional to the ſines of their oppoſite angles.*

From any angle B, draw BD perpendicular to the oppoſite ſide AC; call R the radius, *s*A the ſine of the angle A, and *s*C the ſine of the angle C: then by *Art.* 75, we get,

$$\left.\begin{array}{l} sA : R :: BD : AB \\ sC : R :: BD : BC \end{array}\right\} sA : sC :: BC : AB, \text{ Art. } 24.$$

Caſe

Case I.

82. *Two angles* A, B, *and one side* AC *being given; to find the other sides.*

Let the angle A be 25 degrees, the angle B 80, and the side AC=122: then the sine 98480, of the angle B (80) is to the sine 42261 of the angle A (25), as the side AC (122) is to the side BC= 52.35.

The difference between the sum 105 of the angles A, B, and 180, gives 75 degrees for the angle C: hence the sine 98480 of the angle B (80) is to the sine 96592 of the angle C (75) as the side AC (122) is to the side AB=119.66.

Case II.

83. *Two sides* AB, BC *of a triangle, and one angle* A, *opposite to one of the sides, being given; to find the other side and angles.*

Let AB=100, BC=60, and the angle A 30 degrees: then BC (60) is to AB (100), as the sine 50000 of the angle A (30) is

to the ſine 83333 of the angle C; which by the tables is found to be 56 degrees 27 minutes.

The difference between the ſum 86d. 27m. of the angles A, C, and 180, gives 93d. 33m. for the angle B.

Therefore the ſine 50000 of the angle A (30) is to the ſine 99808 of the angle B, or, of its complement 86d. 27m. to 180 degrees, as the ſide BC (60) is to the ſide AC = 119,7.

N. B. If the angle A oppoſite to the given ſide BC is acute, and BC leſs than AB, it muſt be known whether the angle C is leſs or greater than a right angle, otherwiſe this caſe is indetermined; becauſe the ſine found of this angle is likewiſe that of its complement to 180 degrees.

Caſe III.

84. *Two ſides* AB, AC, *of a triangle, and the included angle* A *being given; to find the other ſide and angles.*

Let AC = 100, AB = 60, and the angle A 53d. 48m; then by *Art.* 80, the ſum

160 of the sides AB, AC, is to their difference 40, as the tangent 197110 of 63d. 6m. half the sum of the opposite angles B, C, is to the tangent 49277 of half their difference: this angle is 26d. 14m. by the tables; which being added to 63d. 6m. half the sum, gives 89d. 20m. for the greatest angle B, and being subtracted from 63d. 6m. gives 36d. 52m. for the least angle C.

Hence, the sine 99992 of the angle B (89d. 20m.) is to the sine 83696 of the angle A (53d. 48m.) as the side AC (100), is to the side BC=80.7.

Case IV.

85. *The three sides of a triangle being given; to find the angles.*

Let the side AC=120, AB=100, and BC=80; then by *Art.* 79, the base AC (120) is to the sum 180 of the sides AB, BC, as their difference 20 is to the sum or difference 30 between the segments AD, DC, of the base, according as the angle C is obtuse or acute; which

being added to the base 120, half of 150 their sum gives 75 for the greatest segment AD.

Hence, in the right angled triangle ADB, the side AD (75) is to the hypothenuse AB (100) as the radius 100000, is to the secant 133333 of the angle A, which by the tables is 41 degrees 25 minutes.

And BC (80) is to AB (100) as the sine 66153 of the angle A (41d. 25m.) is to the sine 82691 of the angle C; which is 55 degrees 47 minutes: and the difference between the sum 97d. 12m. of the angles A, C, and 180d. gives 82 degrees and 48 minutes for the angle B.

These are all the common cases that can happen in the application of Trigonometry, to the finding heights and distances, to surveying and to the determining the angles and lines of a Fortification. In all these cases, it requires no more than to measure a base upon the ground, and to take the adjacent angles with an instrument; the rest is found by one or other of the cases above.

MOTION,

MOTION.

Definitions.

Motion is a continual change of places.

Velocity is that affection of motion, by which a body moves over more or leſs ſpace in equal times.

Uniform motion, is when a body moves over equal ſpaces in equal times.

Accelerated motion, is when the velocity continually increaſes, and retarded, when it continually decreaſes.

Momentum, or quantity of motion, is the power or force, by which a moving body ſtrikes another body at reſt, or any impediment that oppoſes its motion.

Abſolute motion, is the change upon immoveable places; and relative the change upon moveable places.

AXIOM.

Effects are always proportional to their adequate cauſes, that is, a force double produces an effect double; a triple, triple, &c.

UNI-

UNIFORM MOTION.

86. *Spaces moved over with an uniform motion, are as the times employed.*

For a body moves over equal ſpaces in equal times, by definition; it will move over twice the ſpace in twice the time, three times the ſpace in triple the time, and ſo on; that is, the ſpaces moved over are always in proportion to the time elapſed during the motion.

87. *Spaces moved over uniformly in equal times, are as the velocities.*

For a body will in the ſame time move over twice the ſpace with a double velocity, thrice the ſpace with a triple velocity, and ſo on; that is, the ſpaces moved over, will always be in proportion to the velocities.

88. *Spaces moved over uniformly, are as the times and velocities conjointly.*

For ſince ſpaces moved over uniformly, are as the times when the velocities are equal, and are as the velocities when the times

times are equal; but when neither the times nor velocities are equal, they will of consequence be as the times and velocities conjointly.

COMPOUND MOTION.

AXIOM.

Whatever motion, or change of motion is produced in a body, it will be proportional to, and in the direction of the force impressed.

PROP. Fig. 29.

89. *If a body be urged by two forces at the same time, in the direction of, and proportional to the sides of a parallelogram; it will by this compound motion move over the diagonal.*

For while the force impressed in the direction AB, causes the body to move from A towards B, the force impressed in the direction AC, will cause the body

to

to move from A towards C; but the ſpace moved over in the direction AB is to the ſpace moved over in the direction AC, as the force impreſſed in the direction AB, is to the force impreſſed in the direction AC, by the laſt axiom. If from any point E in AB, a line be drawn parallel to AC, meeting the diagonal in F; then becauſe AB : AC or BD :: AE : EF, by *Art*. 27. the ſpace moved over in the direction AB, is to the ſpace moved over in the direction AC, as AE is to EF; it is evident that if AE repreſents the ſpace moved over, in any time, in the direction AB, then will EF repreſent the ſpace moved over at the ſame time in the direction AC; and ſo the body by this compound motion, will be in the diagonal AD; and as this always happens in every particle or inſtant of time from the beginning of the motion; the body, therefore, has moved in the diagonal from the beginning, and was never out of it.

90. Hence, the body moves over the diagonal by the joint forces at the ſame

time,

time, that it would have moved over the ſide AB or AC by the ſingle force acting in that direction.

91. From hence, a ſingle force acting in, and proportional to the diagonal AD, would cauſe the body to move over that line in the ſame time, that the force AB or AC, would move over the ſide AB or AC.

Fig. 30.

92. If from the oppoſite angles B, C, the lines BE, CF, are drawn perpendicular to the diagonal AD; the force impreſſed by AB, in the direction of the diagonal AD, will be expreſſed by AE; and the force impreſſed by AC, in the direction of the diagonal, by AF. For if the parallelograms EH, FG, are compleated; the force impreſſed by the diagonal AB, in the direction AD, is by *Art.* 89, expreſſed by AE, and for the ſame reaſon, the force impreſſed by the diagonal AC, in the direction AD, is expreſſed by AF: and as AB, CD, are equal and parallel, the right angled equiangular

triangles ABE, DCF, are equal in all reſpects; that is, AE=DF, and BE=CF: And ſince the forces expreſſed by BE, CF, or AH, AG, are equal, and act in oppoſite directions, they deſtroy each other: therefore the forces AF and AE, or their equal AD, will be that by which the remaining forces of AB, AC, act in the direction of the diagonal AD.

93. Hence, if ever ſo many forces act upon the ſame body in different directions, the direction in which the body is compelled to move, and the reſulting force in that direction may always be determined.

For let the force, Fig. 31, expreſſed by AB, act in that direction; to find the part which urges the body to move in a given direction BC: draw AD perpendicular to BC produced: then will DB expreſs that force, by *Art.* 92.

MOTION Uniformly Accelerated.

Notwithſtanding that gravity acts differently at different diſtances from the center

center of the earth, yet, as the difference is very ſmall near the ſame point of the ſurface, it is conſidered as conſtant in what follows.

PROP. Fig. 32.

94. *If a body begins to fall freely from reſt, it will acquire velocities proportional to the times elapſed from the beginning of the fall.*

Suppoſe the time divided into very ſmall equal particles or inſtants; then as in each inſtant the force of gravity makes equal impreſſions on the body by ſuppoſition, whatever force the body receives in the firſt inſtant, it muſt receive as much in any other; and ſince the impreſſions remain, the velocity acquired in the firſt inſtant, will be doubled in the ſecond, tripled in the third, quadrupled in the fourth, and ſo on continually thro' every inſtant of the fall: Setting aſide the reſiſtance; the velocity of a falling body from reſt, will conſtantly increaſe in pro-

portion

portion to the time elapſed from the beginning of the fall.

95. Hence, if the line AB expreſſes the time of the fall, BC, perpendicular to AB the velocity acquired in that time: by joining AC, and from any point D in AB, the line DE being drawn parallel to BC; then will DE expreſs the velocity acquired in the time AD: For by the equiangular triangles ABC, ADE, the time AB is to the time AD, as the velocity BC is to the velocity DE.

96. *Spaces deſcended through by a body falling freely from reſt, are as the times and velocities conjointly.*

Let the time AB be divided into very ſmall equal particles, or inſtants: then becauſe the velocities, in theſe very ſmall inſtants of times, may be conſidered as conſtant, ſo that the ſpaces deſcended through in theſe inſtants, will be as the times and velocities, by *Art.* 88: and the ſum of all the rectangles, made by the times and velocities, will be as the ſpace deſcribed by the body from the beginning

beginning of the fall; and when their common difference vaniſhes, the area of the triangle ABC, will, by *Art.* 45, be as the ſpace deſcribed during the time AB.

97. Hence, if the time $AB=t$, the velocity $BC=v$, and the ſpace deſcended thro' z; then the triangle ABC will be equal to $\frac{1}{2}vt=z$, or $2z=vt$: and as the time AB (t) is to the velocity BC (v) in a conſtant ratio, by *Art.* 94; let this ratio be as unity to p; then $v=pt$. Now if we expel t by means of theſe two equations, we get $2pz=vv$, and expelling v, then $2z=ptt$. Conſequently, the ſpace deſcended thro' is, by the equation $2z=vt$, as the time and velocity; by $2pz=vv$, as the ſquare of the velocity; and by $2z=ptt$, as the ſquare of the time.

98. Hence, if the time t, be one ſecond, the equation $2z=ptt$, gives $2z=p$: Which ſhews *that* p *expreſſes double the height fallen thro' in the firſt ſecond.*

N. B. It has been found by experiments, that a body falls thro' a ſpace of 16.1 feet in the firſt ſecond, in the latitude 51d. 32m. of London; therefore,

 $p=$

$p=32.2$ feet in this place; and from hence, all the cases of falling bodies may be solved.

Example. *If the time of a falling body be* 4 *seconds, to find the space descended.*

Hence $t=4$, $p=32.2$, and so $2z=ptt$ gives $2z=32.2\times 16$, or $z=257.6$ feet.

If the space descended through be 144.9 *feet, to find the time of its fall.*

The equation $2z=ptt$, gives $2\times 144.9=32.2\times tt$, or $9=tt$, and $3=t$, seconds.

If the space descended through be 1610 *feet, to find the velocity acquired.*

The equation $2zp=vv$, gives $10368=vv$, whose root gives $v=322$ feet; that is, this velocity will carry a body over a space of 322 feet in a second, if uniformly continued.

99. If the rectangle BF be completed, the space moved over uniformly with the velocity AF or BC, acquired by a falling body in the time AB, will be as the rectangle BF, by *Art.* 88; that is, it will be double the space ABC descended through by a falling body in the same time.

PROP.

PROP. Fig. 32.

100. *If a body be thrown upwards with a velocity acquired in the fall from any height, it will riſe to the ſame height, and in the ſame time of its fall.*

For ſince the action of gravity is conſtant and uniform, in whatever time it generates any velocity in a falling body, it muſt deſtroy in a riſing body the ſame velocity, by its acting in a contrary direction. If then the time AB of the fall be divided into any number of ſmall equal particles or inſtants; then whatever velocity is acquired in the fall, it will be deſtroyed in the riſing body in the ſame time: and therefore, when the body is riſen to the ſame height, from which the body had fallen, the whole velocity acquired in that fall has been deſtroyed in the riſing body in the ſame time.

101. *All bodies falling from reſt, acquire the ſame velocity in the ſame time.*

For if two bodies, one double the weight of the other, are let fall together; then as the heavieſt may be divided into

two, each equal to the lighteſt, theſe three equal bodies will deſcend thro' equal ſpaces in equal times: but as gravity acts upon the particles of matter with the ſame force, whether they are joined together or not; theſe two bodies will deſcend through equal ſpaces in equal times. If two bodies are in any proportion in reſpect to their weight: the heavieſt may be divided in the ſame proportion, and gravity will act alike upon the parts, whether they are joined together or not: therefore all bodies deſcend thro' equal ſpaces in equal times.

This has been proved by experiments, for a feather falls as faſt as gold in an exhauſted receiver.

PROJECTILES.

Notwithſtanding that the reſiſtance of the air acts very ſtrongly upon moving bodies, yet as its eſtimation is ſo very difficult, that it will hardly ever be uſeful in practice, we ſhall here treat only of thoſe in a medium void of all reſiſtance.

PROP. Fig. 33.

102. *If a body be projected from a point* A, *in a given direction* AT, *with a given velocity; it is required to find the curve described by the body.*

At the ſame time that the body is carried uniformly in the direction AT, from A to G, N, T, with the given velocity, it will deſcend through the ſpaces GF, NM, TK, by the force of gravity.

Let the given velocity be - - -	c
The velocity in the curve at M -	v
The time of deſcribing the arc AM	t
The ſine of the angle BAG - -	s
The coſine of that angle - - -	n
The baſe AP - - - - - - - -	x
The perpendicular PM - - -	y

The ſpace AN deſcribed uniformly, is as the time and velocity, that is, ct, by *Art.* 88; and the radius 1 is to the coſine n, as AN or ct is to AP$=x=cnt$; again, the coſine n is to the ſine s, as AP to PN$=\frac{sx}{n}$: and becauſe the ſpaces deſcended thro' by falling bodies from reſt, are as the ſquares of the times, by *Art.* 97, the ſquare

ſquare of one ſecond is to the ſquare tt, as $\frac{1}{2}p$ the ſpace deſcended through in a ſecond, by *Art.* 98, is to the ſpace $NM = \frac{1}{2}ptt$, deſcended thro' in t ſeconds; and ſince $PN - NM = PM$, we get $y = \frac{sx}{n} - \frac{1}{2}ptt$.

If m denotes the height deſcended thro' by a fallen body, ſo as to acquire the given velocity c, then $2mp = cc$, by *Art.* 97; by writing the values of cc and tt into the value of y, we get $y = \frac{sx}{n} - \frac{xx}{4mnn}$ for the equation of the curve required.

103. Hence, if the projection is horizontal, the ſine s of the angle of elevation becomes 0, and y becomes negative, and therefore $y = \frac{xx}{4mnn}$, will be the equation of the curve in this caſe.

104. If $y = 0$, then will $AK = x = 4mns$, and by *Art.* 71, $2ns$ expreſſes the ſine of an angle double the angle of elevation.

105. Hence, the horizontal ranges with the ſame projectile velocity $2m$, are as the ſines $2ns$, of angles double thoſe of elevation.

106. If

106. If the angle of elevation be 45 degrees, the ſine of 90 its double being unity, ſhews that the range is the greateſt poſſible with the ſame projectile velocity.

107. Hence, ſince $2ns = 1$, the horizontal range $AK = 4mns$, becomes $= 2m$, that is, twice the height through which a falling body acquires the projectile velocity: And ſince the ſpaces deſcended through by falling bodies, are as the ſquares of the times, by *Art.* 97, it follows, that the horizontal ranges, when the angle of elevation is 45 degrees, are as the ſquares of the times of the bodies flights.

108. If the angle of elevation is 15 degrees, the ſine of 30 its double will be equal to half the radius: therefore the range at that elevation will be half the range, when the elevation is 45 degrees, with the ſame projectile velocity.

109. There are two angles of elevation, the one as much above 45 degrees, as the other is below, which will throw the body to the ſame diſtance with the ſame

projectile velocity: becauſe the ſine s in one caſe becomes the coſine in the other, and the coſine n becomes the ſine.

110. Since y or PM becomes 0, at A and K; there muſt be one value of y greater than the reſt, which, by *Art.* 57, is found to be when $x = 2mns$; that is when $AB = BK$, by *Art.* 109: and the perpendicular BF will be mss.

Geometrical Construction.

112. On AK erect AL perpendicular, and equal to the height from which a falling body acquires the projectile velocity; deſcribe the ſemi-circumference AEL of a circle, interſecting the given direction AT in E, draw ED perpendicular to AK: then if AK is made equal to 4AD, the point K will determine the horizontal range.

For the right angled triangles ADE, LEA, having the angles at A and L equal, by *Art.* 35, N°. 2, they are equiangular; hence the radius 1 is to the ſine s, as LA m is to $AE = ms$; and the radius 1

is

is to the coſine n, as AE, ms, is to AD$=$ nms; and $4nms=$AK, by *Art.* 104.

When the height LA, and the horizontal range AK are given, by taking AD equal to one fourth of AK, the perpendicular DE to AK, will cut the circumference in the point E, through which the line of direction is to be drawn.

If the line DE cuts the circumference in two points E, e, there will be two angles of elevations EAD, eAD, one as much above as the other is below the angle of 45 degrees, which will determine the ſame horizontal range: if the line DE only touches the circle, the angle of elevation will be 45 degrees, and gives the greateſt range: but if that line neither touches nor cuts the circle, the problem is impoſſible with the ſame projectile force.

113. If the object is above or below the battery as at M, draw AM, and call the tangent of the angle PAM, q; then will the radius 1 be to that tangent q as AP, x, is to PM, $y=qx$; and this value

y wrote into the equation $y = \frac{sx}{n} - \frac{xx}{4mnn}$ gives $\pm q = \frac{s}{n} - \frac{x}{4mnn}$; or if a be the tangent, and r the secant of the angle of elevation, then will $a = \frac{s}{n}$, and $r = \frac{1}{n}$ by trigonometry, and the last equation becomes $x = \overline{a \mp q} \times \frac{4m}{rr}$ in this case. The upper sign of q is to be used when the object is above the level of the battery, and the under one when below.

Practical Rules for Horizontal Ranges.

I. *If the range of a body projected at an angle of* 25 *degrees be* 200 *yards, what will be the range, if the body be projected with the same force at an angle of* 30 *degrees.*

The sine of 50 degrees double of 25, is 76604, and the sine of 60 degrees double of 30, is 86602; then because the horizontal ranges are as the sines of angles double those of elevations, *Art.* 105, we get 76604 : 86602 :: 200 : 226 yards for the range required.

II. *If*

II. *If the range of a body projected at 20 degrees of elevation, is 200 yards, what must the angle of elevation be to carry the body to a distance of 300 yards.*

As the ſine of 40 degrees double of 20 is 64278; we get by the ſame article 200 : 300 :: 64278 : 96417 = to the ſine of the angle double that required, which ſine anſwers to an angle of 74 degrees, 37 minutes, whoſe half gives 37 degrees 18.5 minutes for the required one.

III. *Let the time of a body projected at an angle of 45 degrees, be 12 ſeconds, it is required to find the horizontal range.*

Becauſe theſe ranges are as the ſquares of the times, by *Art.* 108; we have the ſquare of one ſecond, to the ſquare 144 of 12 ſeconds, as the ſpace 16.1 feet deſcended thro' in one ſecond, is to the horizontal range 2318.4 feet, or 772.8 yards.

IV. *Let the angle* PAM, *by which the object* M *is above or below the level* AK *of the battery, be 5 degrees, and the angle of elevation 45 degrees, to find the point* M, *when the height m is 300 yards.*

We

We have $x = a \mp q \times \frac{4m}{rr}$, by *Art.* 113; $a = 1.00000$, $q = .08749$, $rr = 2$, and $a - q = .91251$, when the object is above the level: hence $\frac{4m}{rr} = 600$, and this number multiplied by .91251, gives $x = 547.5$ yards, and $y = qx = 47.9$ yards.

If the object is placed 5 degrees below the level of the battery, and the reſt the ſame as before; then $a + q = 1.08749$, and this number multiplied by 600, gives $x = 652.5$ yards, and $y = 57$ yards.

Theſe are all the caſes that can happen in practice; but it will be moſt convenient to make the firſt tryal with an angle of 15 degrees elevation, in order to know, whether the quantity of powder uſed, will carry the body to the propoſed diſtance, which can never exceed twice the range found by the trial.

FINIS.

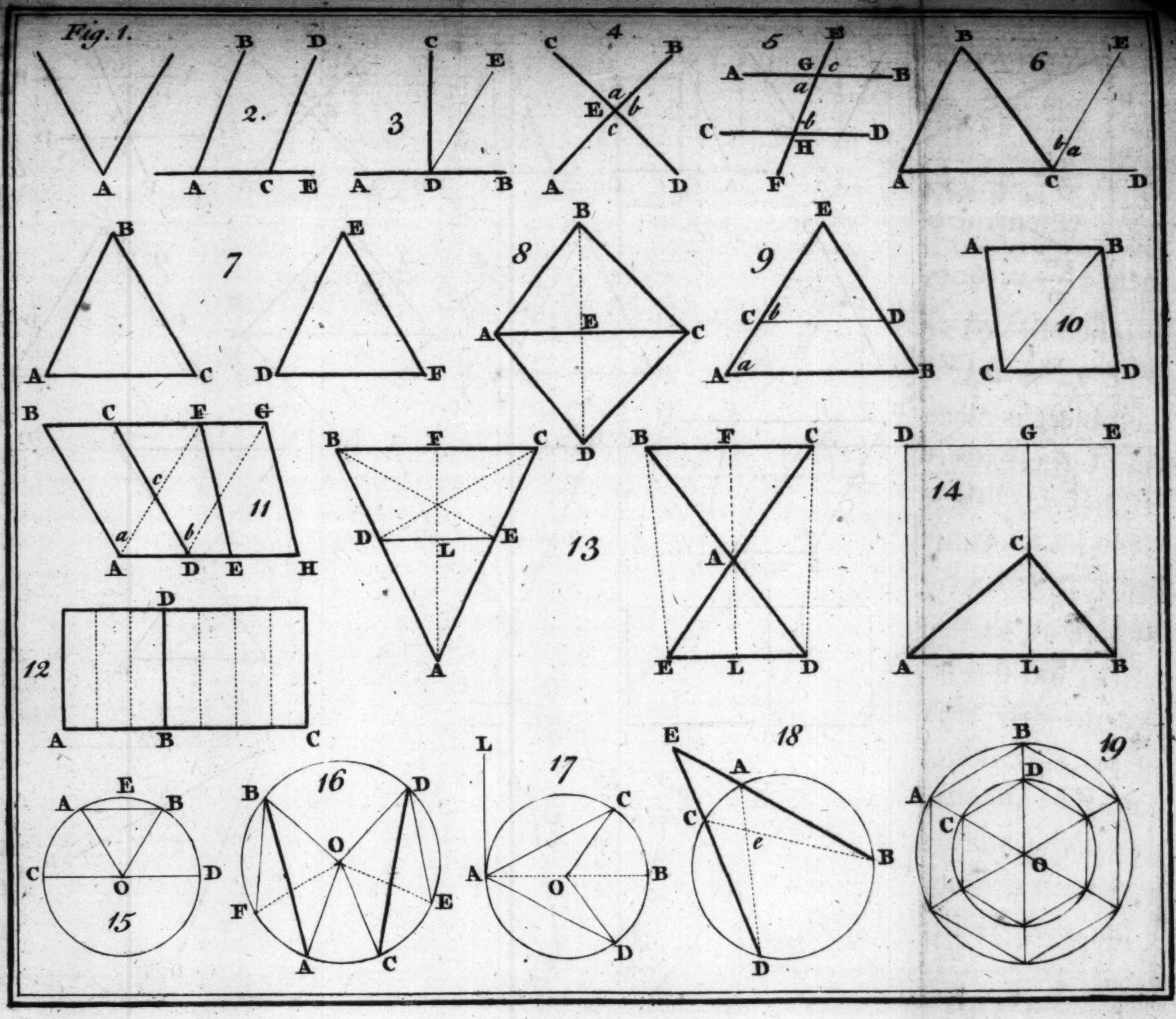
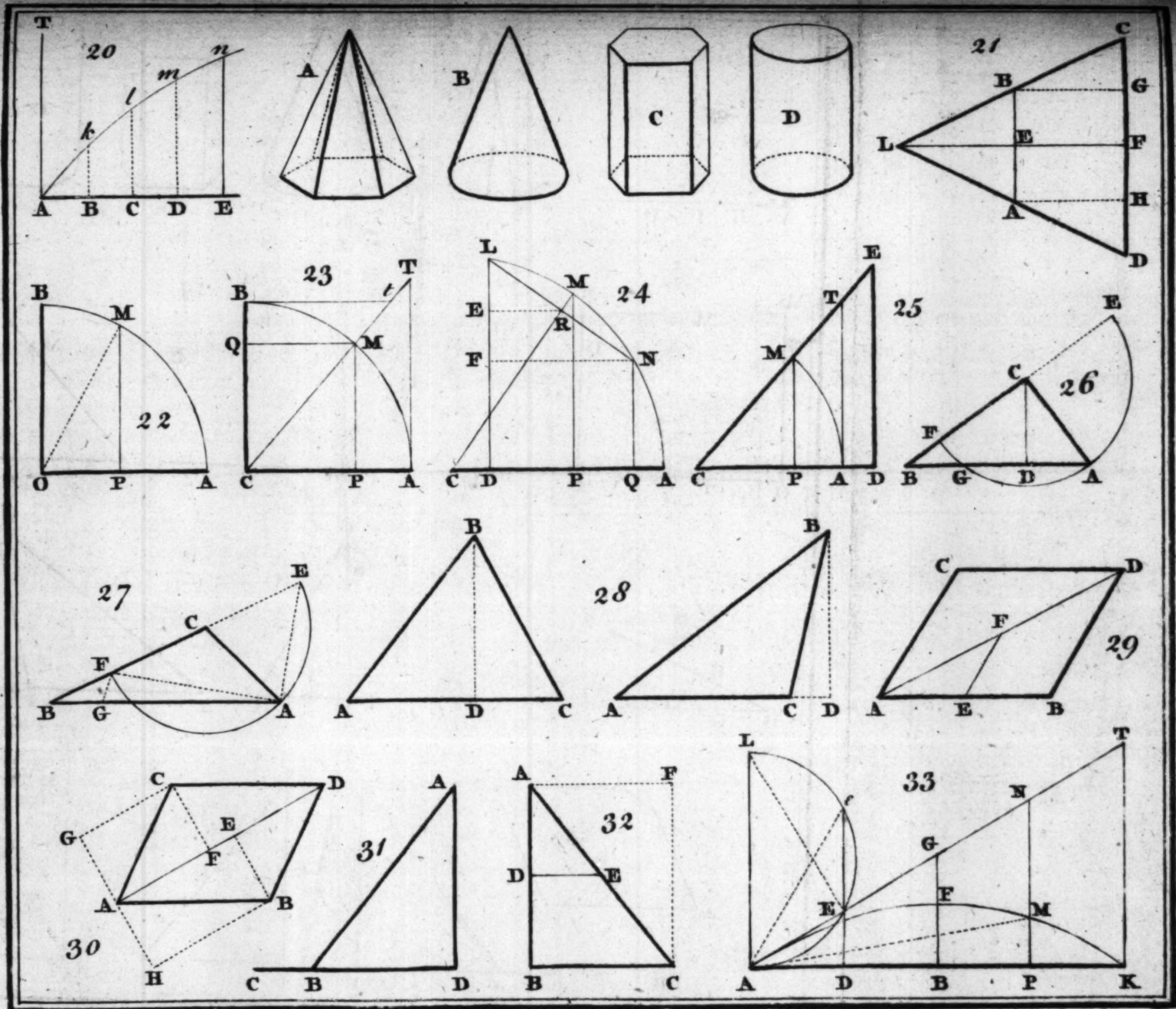


Fig. 1.
A
2.
B D
A C E
3
C E
A D B
4
C B
a
E b
c
A D
5
E
A G c B
a
C b D
H
F
6
B E
b a
A C D
7
B E
A C D F
8
B
A E C
D
9
E
C b D
a
A B
10
A B
C D
11
B C F G
c
a b
A D E H
13
B F C
D L E
A
B F C
A
E L D
14
D G E
C
A L B
12
D
A B C
15
E
A B
C O D
16
B D
O
F E
A C
17
L
C
A O B
D
18
E
A
C
e B
D
19
B
D
A
C
O



20
T k l m n
A B C D E
A
B
C
D
21
C B G E F L H A D
22
B M O P A
23
B T t Q M C P A
24
L M E R F N C D P Q A
25
E T M C P A D
26
E C F B G D A
27
E C F B G A
28
B A D C
B A C D
29
C D F A E B
30
C D G E F A B H
31
A C B D
32
A F D E B C
33
L T t N G E F M A D B P K